Partial Phase Synchronization for Multivariate Synchronizing Systems

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Graphical models applying partial coherence to multivariate time series are a powerful tool to distinguish direct and indirect interdependencies in multivariate linear systems. We carry over the concept of graphical models and partialization analysis to phase signals of nonlinear synchronizing systems. This procedure leads to the partial phase synchronization index which generalizes a bivariate phase synchronization index to the multivariate case and reveals the coupling structure in multivariate synchronizing systems by differentiating direct and indirect interactions. This ensures that no false positive conclusions are drawn concerning the interaction structure in multivariate synchronizing systems. By application to the paradigmatic model of a coupled chaotic Roessler system, the power of the partial phase synchronization index is demonstrated.

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Analyzing interactions between components of a complex network is of genuine interest in various fields of physics. Understanding the characteristic behavior in such networks is of equal importance as the inference of networks' structures from the observation of the networks' behavior. In particular, we address the detection of interdependencies in multivariate dynamic systems.

Various analysis techniques have been suggested to analyze interactions in dynamic systems in diverse fields of research [1–12]. When more than two processes are analyzed, one has to face the problem that complex interaction structures between the processes may arise. For example, two processes in a multivariate system do not have to interact directly. Therefore, bivariate analysis is often not sufficient to reveal the correct interaction structure, i.e., distinguishing direct and indirect interactions. In order to avoid false positive conclusions about the interdependence structure of the investigated multivariate system, the applied analysis technique should distinguish direct and indirect interactions which is impossible for bivariate techniques alone.

For linear systems, the partial spectral coherence was introduced [13] and applied [14,15] to discriminate direct and indirect connections. Graphical models applying partial coherence have been introduced to reveal the interdependence structure in multivariate systems consisting of linear stochastic processes [16]. If a significant bivariate coherence is detected between two processes, which becomes nonsignificant utilizing partial coherence, the corresponding connection is unmasked as an indirect one. An indirect interdependence between two processes, which is mediated by other processes, is therefore distinguishable using partial coherence analysis. As a result of this, false positive conclusions are prevented in linear systems. An intuitively interpretable graph represents the investigated processes as vertices and direct interrelations are represented by edges between the corresponding vertices.

In this Letter, we carry over the concept of graphical models and partialization analysis to nonlinear synchronizing systems. To this aim, considering an N-dimensional dynamic process $X_1, \dots, X_N$, the partial cross spectra $S_{X_k X_l | X_Z}$ between X_k and X_l and the autospectra $S_{X_k X_k | X_Z}$ of X_k conditioning on all remaining processes X_Z $\{X_Z | Z = 1, \dots, N, Z \neq k, l\}$ are defined by

$$S_{X_k X_l | X_Z}(\omega) = S_{X_k X_l}(\omega) - S_{X_k X_Z}(\omega) S^{-1}_{X_Z X_Z}(\omega) S_{X_Z X_l}(\omega) \tag{1}$$

and by

$$S_{X_k X_k | X_Z}(\omega) = S_{X_k X_k}(\omega) - S_{X_k X_Z}(\omega) S^{-1}_{X_Z X_Z}(\omega) S_{X_Z X_k}(\omega). \tag{2}$$

$S_{X_k X_l}(\omega)$, $S_{X_k X_Z}(\omega)$, and $S_{X_Z X_Z}(\omega)$ denote the multivariate autospectra and cross spectra, which can be estimated, e.g., by smoothing the corresponding periodograms

$$\mathrm{Per}_{X_k X_l}(\omega) \propto \sum_t X_k(t) e^{-i\omega t} \sum_t X_l(t) e^{i\omega t} \tag{3}$$

and

$$\mathrm{Per}_{X_k X_k}(\omega) \propto \left| \sum_t X_k(t) e^{-i\omega t} \right|^2 \tag{4}$$

after tapering the time series to avoid misalignment [17].

Inverting and renormalization of the spectral matrix

$$\mathbf{S}(\omega) = \begin{pmatrix} S_{X_1X_1}(\omega) & S_{X_1X_2}(\omega) & \dots & S_{X_1X_N}(\omega) \\ S_{X_2X_1}(\omega) & S_{X_2X_2}(\omega) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ S_{X_NX_1}(\omega) & \dots & \dots & S_{X_NX_N}(\omega) \end{pmatrix} \quad (5)$$

is a numerically efficient method to estimate the partial autospectra and cross spectra $(\mathbf{S}^{-1})_{ij} = \text{const} \times S_{X_iX_j|X_Z}$ [16]. The information about the linear interrelation between the processes X_k and X_l conditioned on the remaining examined processes X_Z is contained in the partial coherence

$$\text{PCoh}_{X_kX_l|X_Z}(\omega) = \frac{|S_{X_kX_l|X_Z}(\omega)|}{\sqrt{S_{X_kX_k|X_Z}(\omega)S_{X_lX_l|X_Z}(\omega)}}. \quad (6)$$

A graphical model reflecting the partial correlation structure consists of a set of vertices $V = 1, \ldots, N$ and a set of edges E, where an edge between k and l is present if $\text{PCoh}_{X_kX_l|X_Z}(\omega)$ is nonzero [16]. We mention that there is a similar concept for ordinary random variables, e.g., covariance selection models [18,19].

For nonlinear systems, in contrast, apart from many other analysis techniques, like for instance bispectral analysis [20,21], the investigations of phase synchronization have gained particular interest [8,9,22–25]. Even for chaotic oscillatory processes and for low coupling strengths between the processes, a synchronization of phase signals has been observed [23].

Similar to the analysis of linear systems, in several applications a network of more than two processes is observed; each pairwise combination has been analyzed separately in order to detect a synchronization in these multivariate nonlinear systems [24,26]. Furthermore, cluster analysis approaches have been proposed to process multivariate data sets [25].

Nevertheless, disentangling directly and indirectly coupled oscillators remains an unsolved problem. If, for instance, two independent oscillators are coupled to one common oscillator, phase synchrony would also be observed between the indirectly coupled oscillators. False positive conclusions about the underlying coupling scheme would be drawn in this example using pairwise phase synchronization analysis.

In the following, we derive a procedure to generalize the concept of partialization analysis to nonlinear synchronizing systems. To this aim, a phase $\Phi(t)$ of a real-valued oscillatory signal $X(t)$ has to be defined leading to

$$V(t) = A(t)e^{i\Phi(t)} = A(t)Q(t). \quad (7)$$

One way, but not the exclusive way, utilizes Gabor's analytic signal approach [27],

$$V(t) = X(t) + iX_h(t) = A(t)e^{i\Phi(t)} = A(t)Q(t), \quad (8)$$

applying the Hilbert transformation of $X(t)$ to obtain the imaginary part $X_h(t)$. Alternative approaches are, for instance, based on wavelet transformations [28,29]. Using the polar representation of $V(t)$, the amplitude $A(t)$ and the phase $\Phi(t)$ are defined. Bivariate synchronization analysis is based on $Q(t)$. We approach partial phase synchronization analysis on the basis of the time series $Q(t)$ by identification of the function $A(t)$ as a taper window for $Q(t)$ and using the concept of partialization similar to the linear theory.

Plugging $Q_k(t) = \exp[i\Phi_k(t)]$ in the Eqs. (3) and (4) of the periodograms leads to

$$\text{Per}_{Q_kQ_l}(\omega) \propto \sum_t Q_k(t)e^{-i\omega t}\sum_t Q_l(t)^* e^{i\omega t} \quad (9)$$

$$= \sum_{t,t'} e^{i(\Phi_k(t)-\Phi_l(t-t'))}e^{-i\omega t'} \quad (10)$$

and

$$\text{Per}_{Q_kQ_k}(\omega) \propto \sum_{t,t'} e^{i(\Phi_k(t)-\Phi_k(t-t'))}e^{-i\omega t'}, \quad (11)$$

respectively.

To introduce the partial phase synchronization index for coupled nonlinear oscillators k and l, the periodogram values of all frequencies are summed up leading to

$$R_{k,l} = d\sum_\omega \text{Per}_{Q_kQ_l}(\omega) = \frac{1}{T}\sum_t e^{i[\Phi_k(t)-\Phi_l(t)]}, \quad (12)$$

with some appropriately chosen constant d ensuring $R_{k,k} = 1$. We emphasize that only the phase differences $[\Phi_k(t) - \Phi_l(t)]$ between the oscillators are contained in the expression for $R_{k,l}$.

The expression (12) is identical to a bivariate phase synchronization index [30]

$$|R_{k,l}^{n,m}| = \left|\frac{1}{T}\sum_{t=1}^{T} e^{i\Phi_{k,l}^{n,m}(t)}\right| \quad (13)$$

for $n = m = 1$. The synchronization index $R_{k,l}^{n,m}$ quantifies the sharpness of peaks in the histograms of the phase difference $\Phi_{k,l}^{n,m} = n\Phi_k(t) - m\Phi_l(t)$ for appropriate integers n and m. It is normalized between zero and one. A value close to 1 is obtained for an almost constant phase difference $|n\Phi_k(t) - m\Phi_l(t)| = |\Phi_{k,l}^{n,m}(t)| < \text{const}, n, m \in \mathbb{Z}$ which has also been observed between two nonidentical chaotic oscillators [23]. For the extension of phase synchronization to nonlinear stochastic oscillators, the distribution of $\Psi_{k,l}^{n,m}(t) = \Phi_{k,l}^{n,m}(t) \bmod 2\pi$ is investigated [8].

For linear systems the autospectra and cross spectra enter the spectral matrix (5) to estimate the partial autospectra and cross spectra [Eq. (1) and (2)] leading to partial coherence [Eq. (6)]. The derivation above proved that for synchronizing systems the spectral matrix (5) has to be substituted by

$$\mathbf{R} = \begin{pmatrix} 1 & R_{1,2} & \dots & R_{1,N} \\ R^*_{1,2} & 1 & \dots & R_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ R^*_{1,N} & R^*_{2,N} & \dots & 1 \end{pmatrix} \tag{14}$$

with entries $R_{k,l} := R^{n,m}_{k,l}$ [Eq. (13)], which are the pairwise synchronization indices. The asterisk denotes complex conjugation. In the following we refer to this matrix as synchronization matrix. Caused by the proved analogy between spectral and synchronization theory, the inverse $\mathbf{PR} = \mathbf{R}^{-1}$ of the synchronization matrix $\mathbf{R}$ immediately leads to the definition of the $n{:}m$ partial phase synchronization index

$$R_{k,l|Z} = \frac{|\mathbf{PR}_{kl}|}{\sqrt{\mathbf{PR}_{kk}\mathbf{PR}_{ll}}} \tag{15}$$

between X_k and X_l conditioned on the remaining processes $\{X_Z | Z = 1, \dots, N, Z \neq k, l\}$. It replaces the partial coherence [Eq. (6)] for synchronizing systems. As for partial coherence, where the indirect interactions are characterized by an absent partial coherence accompanied by a bivariate significant coherence [16], the following holds: If the bivariate phase synchronization index $R_{k,l}$ is considerably different from zero, while the corresponding multivariate partial phase synchronization index $R_{k,l|Z} \approx 0$, there is strong evidence for an indirect coupling between the processes X_k and X_l. Graphical models applying partial phase synchronization analysis are defined by the following:

An edge E between the oscillators k and l in a partial phase synchronization graph is missing, if and only if $R_{k,l|Z}$ is small compared to $R_{k,l}$.

Three coupled stochastic Roessler oscillators

$$\dot{\xi}_j = \begin{pmatrix} \dot{X}_j \\ \dot{Y}_j \\ \dot{Z}_j \end{pmatrix} = \begin{pmatrix} -\omega_j Y_j - Z_j + [\sum_{i\neq j} \varepsilon_{i,j}(X_i - X_j)] + \sigma_j \eta_j \\ \omega_j X_j + a Y_j \\ b + (X_j - c) Z_j \end{pmatrix} \qquad i, j = 1, 2, 3 \tag{16}$$

are a genuine example of a system consisting of weakly coupled self-sustained stochastic oscillators. The parameters are set to $a = 0.15$, $b = 0.2$, $c = 10$, $\omega_1 = 1.03$, $\omega_2 = 1.01$, and $\omega_3 = 0.99$ yielding a chaotic behavior in the deterministic case. For the noise term $\sigma_j \eta_j$ a standard deviation of $\sigma_j = 1.5$ is chosen and η_j is standard Gaussian distributed. Both the bidirectional coupling $\varepsilon_{1,3} = \varepsilon_{3,1}$ between oscillator ξ_1 and oscillator ξ_3, and the bidirectional coupling $\varepsilon_{1,2} = \varepsilon_{2,1}$ between oscillator ξ_1 and oscillator ξ_2 are varied between 0 and 0.3. Both synchronization phenomena, phase and lag synchronization, are contained in this range of coupling strengths. The oscillators ξ_2 and ξ_3 are not directly coupled since $\varepsilon_{2,3} = \varepsilon_{3,2} = 0$. The coupling scheme is summarized in Fig. 1(a).

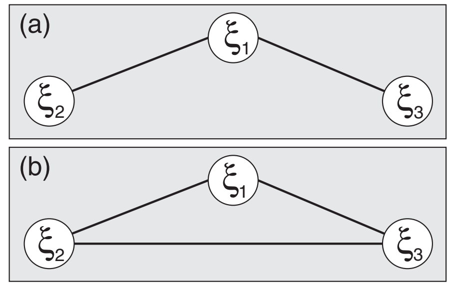

FIG. 1. (a) Graph for the simulated coupling scheme in the Roessler system. The direct coupling between oscillators ξ_2 and ξ_3 is absent. (b) Graph based on bivariate synchronization analysis. An additional but spurious edge between oscillator ξ_2 and ξ_3 is present.

In the following an example of 1:1 synchronization of the X components is investigated. The bivariate synchronization index $R_{1,2}$ as well as $R_{1,3}$ increases when the corresponding coupling strength is increased, indicating phase synchronization (Fig. 2 upper row). Once a sufficient amount of coupling exists between oscillators ξ_1 and ξ_2 as well as between ξ_1 and ξ_3, a nonvanishing bivariate synchronization index $R_{2,3}$ between the not directly coupled

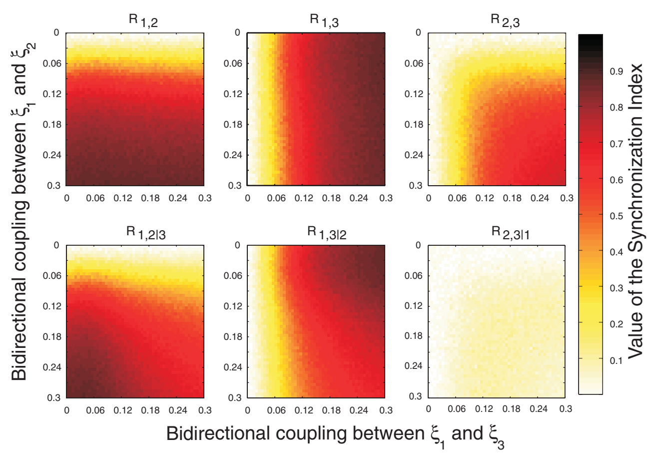

FIG. 2 (color online). The partial phase synchronization index. Coupling strengths between oscillators ξ_1 and ξ_2 and between oscillators ξ_1 and ξ_3 are varied between 0 and 0.3, for an absent coupling between ξ_2 and ξ_3. Values of the bivariate phase synchronization index (upper row) and partial phase synchronization index (lower row) are shown. When comparing the bivariate phase synchronization index $R_{2,3}$ with the partial phase synchronization index $R_{2,3|1}$ it becomes clear that the interaction between oscillator ξ_2 and ξ_3 is mediated by ξ_1 since $R_{2,3} \gg R_{2,3|1}$.

oscillators ξ_2 and ξ_3 is observed (Fig. 2 upper row). This high but spurious phase synchronization is caused by the common influence from oscillator ξ_1 onto ξ_2 and ξ_3. The bivariate synchronization analysis suggests the coupling scheme between the three Roessler oscillators summarized in Fig. 1(b), containing the additional but spurious edge between oscillator ξ_2 and ξ_3.

In Fig. 2 (lower row) the results of partial phase synchronization analysis are shown. While $R_{1,2|3}$ as well as $R_{1,3|2}$ are essentially unchanged compared to the bivariate synchronization indices, $R_{2,3|1}$ stays almost always below 0.1 and is therefore considerably smaller than $R_{2,3}$ in the area of spurious synchronization. This strongly indicates the absence of a direct coupling between oscillators ξ_2 and ξ_3. This results in the graph presented in Fig. 1(a), representing the correct coupling scheme.

The proposed approach relies on the extracted phase. A definition of the phase signal is possible whenever the analytic signal circulates around the phase space origin as it the case in many applications such as cardiorespiratory systems or systems investigated in neuroscience, geoscience, etc. In those cases, where an application of an analytic signal is not possible, but where it is nonetheless possible to motivate concepts like phase synchronization by means of, for instance, recurrences, further investigation is inevitable [31].

In summary, we introduced graphical models applying partial phase synchronization to phase signals of multivariate synchronizing oscillators. This multivariate extension is essential when analyzing multivariate systems to avoid false positive conclusions about the underlying coupling scheme. To this aim, we derived an index quantifying synchronization by applying a partialization analysis to the analytic signal of the oscillators, which also gives a firm mathematical basis for the often applied but only heuristically introduced synchronization index. Inverting the synchronization matrix yields the concept of partial phase synchronization, that has been shown to differentiate direct and indirect coupling in a multivariate system of nonlinear synchronizing oscillators. In a forthcoming paper, we will present the applicability of this novel approach to empirical data.

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