Mixing properties of the Rössler system and consequences for coherence and synchronization analysis

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Cross-spectral and synchronization analysis of two independent, identical chaotic Rössler systems suggest a coupling although there is no interaction. This spuriously detected interaction can either be explained by the absence of mixing or by finite size effects. To decide which alternative holds the phase dynamics is studied by a model of the fluctuations derived from the system's equations. The basic assumption of the model is a diffusive character for the system which corresponds to mixing. Comparison of theoretical properties of the model with empirical properties of the Rössler system suggests that the system is mixing but the rate of mixing appears to be rather low.

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I. INTRODUCTION

Beside ergodicity, mixing is the most important stochastic feature of chaotic systems which is essential for cross-spectral and synchronization analysis on the basis of measured data. This is because the asymptotic distribution of these methods relies on the validity of the central limit theorem and the asymptotical independence of time lagged events [1]. Strong mixing, which also implies the decay of the autocovariance function, turns out to satisfy these requirements and is therefore the suitable definition for our purposes. Let $M_t(x)$ be the time evolution of a certain point x in phase space by an ergodic dynamic system. The invariant measure is denoted by μ. Strong mixing of this dynamical system is then satisfied if for all μ-measurable sets A, B the condition

$$\lim_{t\to\infty} \mu[A \cap M_t^{-1}(B)] = \mu(A)\mu(B) \tag{1}$$

is valid [3]. The set $M_t^{-1}(B)$ in Eq. (1) is thought to be a compact notation for the inverse image of M_t, $M_t^{-1}(B)=\{x: M_t(x)\in B\}$. Using the definition above, the system of our interest, namely the Rössler system [2], $dx/dt=-y-z$, $dy/dt=x+ay$, $dz/dt=b+(x-c)z$, shows for a specific set of the parameters $a=b=0.2$, $c=6.3$ a behavior which could be explained by a defect of mixing. An alternative explanation would be the presence of finite size effects. This paper is entitled to discriminate these alternatives.

A possible loss of mixing is connected to the nonhyperbolicity of the system, since for hyperbolic or axiom A systems mixing is always satisfied. Furthermore, the mixing coefficient, describing the statistical dependency of time lagged events or the correlations of sufficiently smooth observables is decaying exponentially and the rate of this decay is related to the positive Lyapunov exponents. The key point of these statements is the qualitative knowledge of the spectrum of the time evolution operator for the system density P_t, the so-called Frobenius-Perron operator (FPO). Note that the system density is chosen to be absolutely continuous with respect to some invariant measure. It can be shown that the resolvent function of the FPO $R(z)=(\mathbb{1}z-P_t)^{-1}$ can be meromorphically extended onto the whole complex space [4–6]. The poles of the resolvent function are lying in the interior of the unit circle except for the simple pole at one, corresponding to the invariant measure. Since poles of the resolvent function are the point spectrum of the FPO and by using $P_{n\Delta t}=P_{\Delta t}^n$, the exponential decay of correlations for sufficiently smooth real-valued observables can be shown. Moreover, eigenvalues close to the unit circle are generating sharp peaks of approximately Lorentzian shape in the power spectrum. This consequence is in perfect accordance with the more heuristic derivation of peak shapes of chaotic oscillators given in Ref. [7]. The corresponding eigenvalues are called Ruelle-Pollicott resonances.

In case of nonhyperbolic systems the discussed properties of the resolvent function need not be fulfilled. Generally, the Lyapunov exponents are not related to the rate of mixing, even if the process of interest satisfies condition (1). Instead, the spectrum of the FPO may have a cluster point on the unit circle which leads to a loss in mixing. Nonrigorous methods such as calculating the spectrum of the FPO in a finite dimensional approximation and performing the limit of infinite dimension have been applied in Refs. [8,9]. The comparison of the analytically derived results are in good accordance with simulations, even though there is no rigorous justification of this method.

For the Rössler system such a procedure is not feasible, since the FPO P_t can only be approximated numerically and thus the limit of infinite dimension is not possible. If the last step is omitted, the calculated eigenvalues and eigenfunctions would depend on the chosen set of basis functions. Inconsistency would therefore be the consequence of such a procedure. It is therefore not likely to approach the question of mixing of the Rössler system on the basis of the FPO.

It turns out that the crucial point of mixing for the Rössler system is the dynamics of the phase in the x-y plane. Before

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analyzing the phase dynamics in detail, consequences resulting from the absence of mixing are reviewed and empirical results are given for the power spectrum, and cross-spectral analysis in Sec. II, as well as synchronization analysis in Sec. III. A detailed analysis of the phase dynamics is then given in Sec. IV.

II. SPECTRAL AND CROSS-SPECTRAL ANALYSIS AND MIXING

The eigenvalue spectrum of the FPO determines the power spectrum of dynamical systems. To demonstrate this statement, let f, g be real valued observables satisfying

$$\lim_{n\to\infty} P^n_{\Delta t} f = \lim_{n\to\infty} P^n_{\Delta t} g = 0. \tag{2}$$

The dynamical system is assumed to be ergodic, therefore the unique invariant measure μ exists and its density corresponds to the nondegenerate eigenvalue 1 of the FPO. Due to Eq. (2), the observables f, g are orthogonal to the eigenspace, in which the invariant density lies. The correlation function is then

$$C^{f,g}(n) = \int g(x) P^n_{\Delta t} f(x) \mu(dx).$$

For some complex z satisfying $|z|>1$, the Laplace transformation of this correlation function

$$\tilde{C}^{f,g}(z) = \sum_{n=0}^{\infty} C^{f,g}(n) z^{-n}$$

can be rewritten in terms of the resolvent function $R(z)$ of the FPO and yields

$$\tilde{C}^{f,g}(z) = \int g(x) z R(z) f(x) \mu(dx). \tag{3}$$

Suppose that f can be decomposed into eigenfunctions f_i of the FPO, $f(x)=\sum_{i=1}^{\infty} a_i f_i(x)$. The eigenvalues of f_i are denoted by z_i and are satisfying $|z_i|\leqslant 1$ since $P_{\Delta t}$ is a Markov operator. Equation (3) then yields

$$\tilde{C}^{f,g}(z) = \sum_{i=1}^{\infty} \frac{a_i z}{z - z_i} \int g f_i d\mu. \tag{4}$$

If all eigenvalues z_i are compactly contained in the unit circle, Eq. (4) is defined for $z=e^{i\omega\Delta t}$. Thus, $S(\omega)=\tilde{C}^{f,g}(e^{i\omega\Delta t})$ is the one-sided Fourier transform of the correlation function or the power spectrum. The power spectrum is usually defined by the two-sided Fourier transformation but in the case of noninvertible dynamics such a power spectrum would not be defined. Therefore, the general structure of such a power spectrum is given by the smooth function

$$S(\omega) = \sum_{j=1}^{\infty} \frac{\gamma_j e^{i\omega\Delta t}}{e^{i\omega\Delta t} - z_j},$$

where eigenvalues close to the unit circle are able to produce resonances of Lorentzian shape. Such a specific distribution of the eigenvalues corresponds to a dynamical system equipped with the mixing property.

In the case of the absence of mixing, in which the eigenvalues z_i are having a cluster point on the unit circle, the transition from $|z|>1$ to $|z|=1$ in Eq. (4) is not possible. But due to the assumed ergodicity the correlation function exists and thus the Wiener-Khintchine theorem guarantees the existence of a spectral distribution function [12]. Such a distribution function is in general not represented by a smooth density, instead delta distributions are often present.

For empirical time series of length N, the power spectrum can be estimated by calculating the discrete Fourier transform of the observed time series. The squared norm of the Fourier transform then defines the periodogram

$$\mathrm{Per}(\omega) = |f(\omega)|^2, \quad f(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} x(t) e^{-i\omega t}.$$

If the power spectrum is a smooth function in the frequency domain and the time series mixes sufficiently, the periodogram $\mathrm{Per}(\omega)$ is χ^2-distributed,

$$\mathrm{Per}(\omega) \sim S(\omega)\chi^2_2/2, \quad \omega \neq 0, \pi,$$

which is due to the central limit theorem [10,11,16]. Increasing N, increases the frequency resolution but does not reduce the variance of the periodogram. In order to obtain a consistent estimation of the power spectrum, in which the variance vanish if $N\to\infty$, the periodogram must be smoothed [10,12]. If the spectral density, e.g., contains a delta distribution the smoothing procedure is no longer consistent. Due to the orthogonality of the Fourier transform, the growth of height with respect to the amount of data for this component is proportional to N.

The x component of the discussed Rössler system is showing sharp peaks in the power spectrum, Fig. 1(a). By increasing the amount of data N, the peak seems to grow in height but the growth rate cannot be determined because of finite size effects. This is mainly due to the uncertainty of the peak location and truncation effects, also known as tapering effects.

An analysis technique for detecting linear relationship between two processes $x(t)$ and $y(t)$ is cross-spectral analysis. The processes x, y are assumed to have zero mean and unit variance, if not a linear transformation must be applied such that the processes are satisfying these requirements. The cross spectrum is then defined by the Fourier transformation of the cross-correlation function,

$$\mathrm{CCF}(\tau) = \langle x(t) y(t-\tau) \rangle,$$

$$\widehat{\mathrm{CCF}}(\omega) = \frac{1}{2\pi} \sum_{\tau} \mathrm{CCF}(\tau) \exp(-i\omega\tau),$$

normalized by the product of square root of the univariate power spectra $S_x(\omega)$, $S_y(\omega)$,

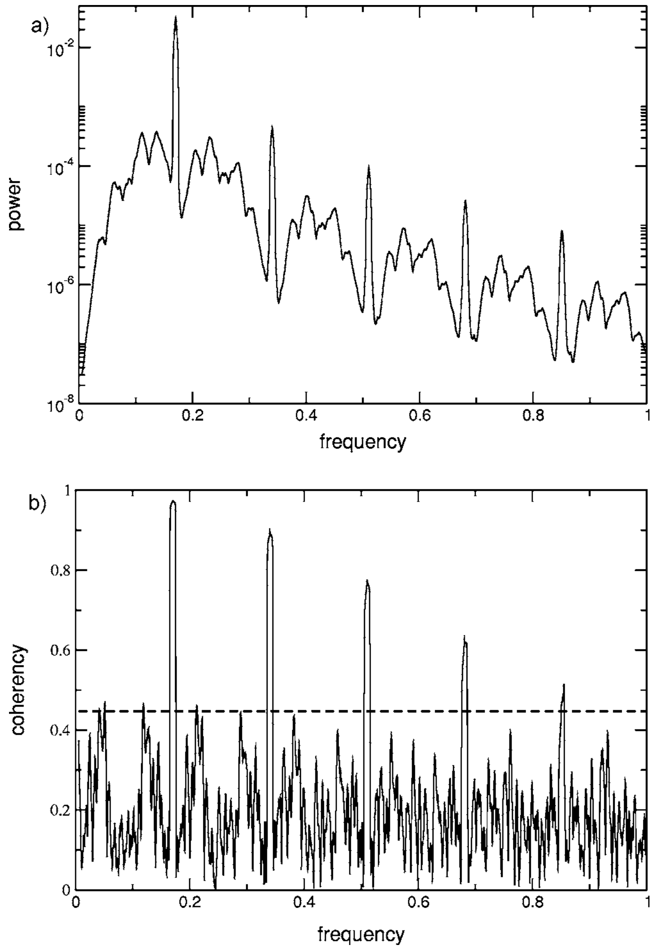

FIG. 1. (a) Power spectrum of the x component of the Rössler system at the vicinity of the main oscillating frequency. (b) Coherency of two independent, identical Rössler systems. The coherency at frequencies of approximately 0.17 and its multiples are lying above the 5% level of significance (dashed line).

$$\mathrm{CS}_{xy}(\omega) = \frac{\widehat{\mathrm{CCF}}(\omega)}{\sqrt{S_x(\omega)S_y(\omega)}}.$$

The brackets $\langle\ \rangle$ above denote the expectation value. This function is in the general complex and can therefore be decomposed into the phase spectrum $\Phi_{xy}(\omega)$ and the coherency $\mathrm{Coh}_{xy}(\omega)$, such that

$$\mathrm{CS}_{xy}(\omega) = \mathrm{Coh}_{xy}(\omega)e^{i\Phi_{xy}(\omega)}.$$

Due to the normalization of the cross spectrum the coherency is ranging from $\mathrm{Coh}_{xy}(\omega)=0$, no linear relationship between x and y at ω, to $\mathrm{Coh}_{xy}(\omega)=1$, perfect linear relationship. Whereas the interpretation of the phase spectrum $\Phi_{xy}(\omega)$ is more difficult, see, e.g., Ref. [13]. Since we are only interested in detecting the presence of an interaction between x and y a deeper discussion of the phase spectrum is not needed.

The estimation of the cross spectrum is analogous to estimation of the power spectrum. Furthermore, an asymptotic level of significance under the hypothesis $\mathrm{Coh}_{xy}(\omega)=0$ can be derived

$$s = \sqrt{1-\alpha^{2/(\nu-2)}}, \tag{5}$$

where ν is the number of equivalent degrees of freedom depending on the smoothing procedure of the spectra [10–15] and the rejection probability α. The value of $\alpha \in [0,1]$ is the probability that even if $\mathrm{Coh}_{xy}(\omega)=0$ the hypothesis is rejected on the basis of the observed data, which is due to random fluctuations of the estimated coherency. Only such a level of significance allows to decide whether the observed coherency is different from zero, and is therefore extremely important for the following argumentation.

Mixing of the processes is again essential for cross-spectral analysis and for deriving Eq. (5). In case of two independent causal processes x and y we obviously have $\mathrm{Coh}_{xy}=0$. According to Refs. [10,11], the estimation of this quantity is possible if the processes can be approximated by the linear sequences

$$x(t) = \sum_{i=0}^{\infty} C_1(i)z_1(t-i) \quad \text{and} \quad y(t) = \sum_{i=0}^{\infty} C_2(i)z_2(t-i),$$

where $z_i(t)$ is an independent and identically distributed sequence of random variables having zero mean and a finite fourth moment. Moreover, the coefficients must satisfy

$$\sum_{j=0}^{\infty} |C_i(j)|j^{1/2} < \infty, \quad i=1,2. \tag{6}$$

Necessarily, $C_i(j) \to 0$ if $j \to \infty$ and hence the autocorrelation of x and y must decay, which is valid if both processes are mixing. Besides the pure estimation of the cross spectrum, statistical inference such as Eq. (5) is based on the asymptotical normality of sums of state variables. Here, mixing is again a central requirement, see e.g., Ref. [1].

Now, two independent, identical Rössler systems of length 5×10^5 data points are simulated by using randomly generated initial conditions. For the following simulations, the Rössler system is integrated by a Runge-Kutta scheme of fourth order with step size control keeping the numerical error below $\epsilon=10^{-12}$ [16]. The sampling rate of both time series was chosen to be $\Delta t=0.01$. If the conditions of the cross-spectral analysis are valid, coherency of the x component should be zero, since there is no (linear) relationship between the time series. But Fig. 1(b) clearly shows a significant coherency. This result can be interpreted in two different ways, (1) mixing is violated as outlined above or (2) the decay of the phase correlations is too slow such that the cross-spectral analysis has not reached its asymptotic accuracy.

III. SYNCHRONIZATION ANALYSIS AND MIXING

In order to detect a possible phase synchronization between two coupled, oscillatory systems a suitable definition of phase and amplitude of a real-valued observed signal is required. This can be realized, if the considered oscillations are having a narrow frequency band [17,18]. Let $x(t)$ be the

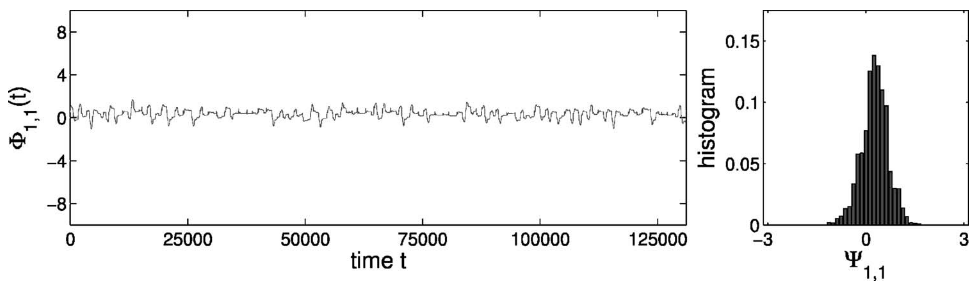

FIG. 2. Time evolution of $\Phi_{1,1}$ for two independent Rössler systems, left graph. The distribution of $\Psi_{1,1}$, right figure, is showing a sharp peak.

real-valued signal satisfying the mentioned property. The analytic signal is then given by

$$\psi(t) = x(t) + i\hat{x}(t) = A(t)e^{i\varphi(t)},$$

where $A(t)$ is the amplitude and $\varphi(t)$ the phase. The imaginary counterpart of the analytic signal can be obtained by the Hilbert transform [19]

$$\hat{x}(s) = \pi^{-1}\mathrm{P.V.}\int \frac{x(t)}{s-t}dt$$

of the signal, in which P.V. refers to Cauchy's principle value. The phase $\varphi(t)$ is now a suitable basis for the synchronization analysis.

Phase synchronization of two coupled, chaotic oscillators occurs if the $n{:}m$ phase locking condition is satisfied [20],

$$|n\varphi_x(t) - m\varphi_y(t)| = |\Phi_{n,m}| < \text{const},$$

where $\varphi_x(t)$, $\varphi_y(t)$ denotes the phase of the time series $x(t)$, $y(t)$ and n, m are given integers. To suppress phase jumps, induced by the presence of numerical or observational noise, $\Phi_{n,m}$ is modified by

$$\Psi_{n,m} = \Phi_{n,m} \bmod 2\pi.$$

The distribution of $\Psi_{n,m}$ then exhibits a sharp peak, if the two oscillators are phase synchronizing [21]. A commonly used quantity, measuring the sharpness of the distribution of $\Psi_{n,m}$ is the synchronization index [22]

$$R_{n,m}^2 = \langle\cos(\Psi_{n,m})\rangle^2 + \langle\sin(\Psi_{n,m})\rangle^2,$$

where $\langle\ \rangle$ denotes the expectation value with respect to the distribution of $\Psi_{n,m}$. The synchronization index is $R_{n,m}=1$ for a constant phase difference between the two time series and $R_{n,m}=0$ for a uniformly distributed phase difference. Note that the usage of the Hilbert transform for determining the phase is the most general approach. In our case the phase can be calculated directly from the *x*-*y* projection, but the outcome of the synchronization analysis does not alter if either the Hilbert transform or the direct computation is used.

The mixing property for the phases is again essential to determine whether the processes are phase synchronizing on the basis of measured data or not. For demonstrating this statement, let us consider two ergodic self-oscillatory systems satisfying $\langle\cos(\Psi_{n,m})\rangle = \langle\sin(\Psi_{n,m})\rangle = 0$, and thus $R_{m,n}^2=0$. Suppose that Ψ_i, $i=1,\ldots,n$ is a suitable realization of $\Psi_{n,m}$ which is equidistantly sampled in t. By the ergodic theorem $R_{n,m}^2$ is given by

$$R_{n,m}^2 = \lim_{N\to\infty}\left[\left(N^{-1}\sum_{i=1}^{N}\sin(\Psi_i)\right)^2 + \left(N^{-1}\sum_{i=1}^{N}\cos(\Psi_i)\right)^2\right]$$
$$= \lim_{N\to\infty} N^{-2}\sum_{i,j=1}^{N}[\sin(\Psi_i)\sin(\Psi_j) + \cos(\Psi_i)\cos(\Psi_j)]. \tag{7}$$

Since the sample is equidistant in time and by using the ergodicity again we have

$$\frac{1}{N-j}\sum_{i=1}^{N-j}[\sin(\Psi_i)\sin(\Psi_{i+j}) + \cos(\Psi_i)\cos(\Psi_{i+j})]$$
$$= \langle\sin(\Psi_1)\sin(\Psi_{1+j}) + \cos(\Psi_1)\cos(\Psi_{1+j})\rangle + r_{Nj} = \xi_j + r_{Nj}, \tag{8}$$

for each $0<j<n$. The remainder r_{Nj} vanishes asymptotically, $\lim_{N\to\infty} r_{Nj}=0$. Inserting Eq. (8) into Eq. (7) we arrive at

$$R_{n,m}^2 = \lim_{N\to\infty}\left(N^{-1} + 2\sum_{j=1}^{N-1}\frac{N-j}{N^2}(\xi_j + r_{Nj})\right) = \lim_{N\to\infty} 2\sum_{j=1}^{N-1}\frac{N-j}{N^2}\xi_j. \tag{9}$$

A necessary condition that $R_{n,m}$ in Eq. (9) vanishes is therefore $\xi_j\to 0$ if $j\to\infty$. Now, consider $\sin(\Psi_{n,m})$ and $\cos(\Psi_{n,m})$ as observables of the processes, then ξ_j is the sum of the autocovariance function of these quantities. Again the autocovariance function asymptotically vanishes if the strong mixing, Eq. (1), is satisfied, the necessary condition is therefore met if both processes are mixing. It should further be noted that the equidistant sampling is not explicitly needed and was only introduced to avoid a rather clumsy notation.

Again, two independent, identical Rössler systems are generated numerically, where the sampling is chosen to be $\Delta t=0.01$ for both realizations of length 131 072. The time evolution of $\Phi_{1,1}$ and the distribution of $\Psi_{1,1}$ is shown in Fig. 2 and reveals that the phase-locking condition seems to be satisfied. Furthermore, the narrow peak of the distribution of $\Psi_{1,1}$ indicates that the synchronization index should be close to unity. Calculating the synchronization index yields $R_{1,1}=0.92$. On the basis of empirical data, one would draw the conclusion that these two time series are phase synchronized which is again spurious, either due to a loss in mixing or due to finite size effects. In addition, these results show that this question can be approached only by analyzing the phase evolution of the Rössler system.

IV. A MODEL OF THE PHASE FLUCTUATIONS

In the following, a model of the phase fluctuations is derived. The analysis shows that the diffusion constant of the Rössler system depends mainly on the inverse square of the amplitude in the *x*-*y* plane. The possibility of such a phase-amplitude dependency of chaotic oscillators is briefly discussed in Ref. [23]. Assuming that the system behaves like a diffusion process and that the z component can be treated as being constant for a given time step $\Delta t \ll 1$ to give an effective approximation of the phase fluctuations. The differential equation then reduces to the form $dx/dt=-y-z_t$, $dy/dt=x+ay$ and can be integrated one step ahead,

$$x_{t+\Delta t} = A_t e^{a\Delta t/2}\cos(\omega\Delta t + \phi_t) - z_t\Delta t,$$

$$y_{t+\Delta t} = A_t e^{a\Delta t/2}[\omega \sin(\omega\Delta t + \phi_t) - \sqrt{1-\omega^2}\cos(\omega\Delta t + \phi_t)], \tag{10}$$

where $\omega^2 = 1-(a/2)^2$ and A_t is the amplitude and ϕ_t is the phase at time t. In order to include the diffusion, Eq. (10) is perturbated by Gaussian white noise and A_t, ϕ_t, z_t are exchanged by their mean values. Setting $\tau=\omega\Delta t+\phi_t$, the extended Eq. (10) yields

$$x_{t+\Delta t} = A_t e^{a\Delta t/2}\cos(\tau) - z_t\Delta t + \sqrt{D_x\Delta t}\epsilon_t,$$

$$y_{t+\Delta t} = A_t e^{a\Delta t/2}[\omega \sin(\tau) - \sqrt{1-\omega^2}\cos(\tau)] + \sqrt{D_y\Delta t}\eta_t,$$

where ϵ_t, η_t denotes uncorrelated white noise and D_x, D_y are the assumed diffusion constants for the x and y component, respectively. Now, the phase $\phi_{t+\Delta t}=\arctan(y_{t+\Delta t}/x_{t+\Delta t})$ is calculated up to order $\sqrt{\Delta t}$ in all noise terms and yields

$$\phi_{t+\Delta t} = \arctan(\kappa_t) + \frac{e^{-a\Delta t/2}}{A_t\cos(\tau)(1+\kappa_t^2)}(\kappa_t z_t\Delta t + \sqrt{D_y\Delta t}\eta_t - \kappa_t\sqrt{D_x\Delta t}\epsilon_t) + \mathcal{O}(\Delta t),$$

where $\kappa_t=\omega\tan(\phi_t)-\sqrt{1-\omega^2}$. The diffusion constant of the phase is therefore determined by $D_{A_t,\phi_t}=\lim_{\Delta t\to 0}\mathrm{Var}(\phi_{t+\Delta t})/\Delta t$, where Var denotes the variance of a random variable. Since $\lim_{\Delta t\to 0}\tau=\lim_{\Delta t\to 0}(\omega\Delta t+\phi_t)=\phi_t$,

$$D_{A_t,\phi_t} = \frac{1}{A_t^2\cos^2(\phi_t)(1+\kappa_t^2)^2}(\kappa_t^2 D_x + D_y). \tag{11}$$

If $a \ll 1$ then $\omega\approx 1$ and thus Eq. (11) reduces to

$$D_{A_t,\phi_t} \approx \frac{\sin^2(\phi_t)D_x + \cos^2(\phi_t)D_y}{A_t^2}.$$

The variance of the system's phase $\varphi(t)$ at time t can then be approximated by

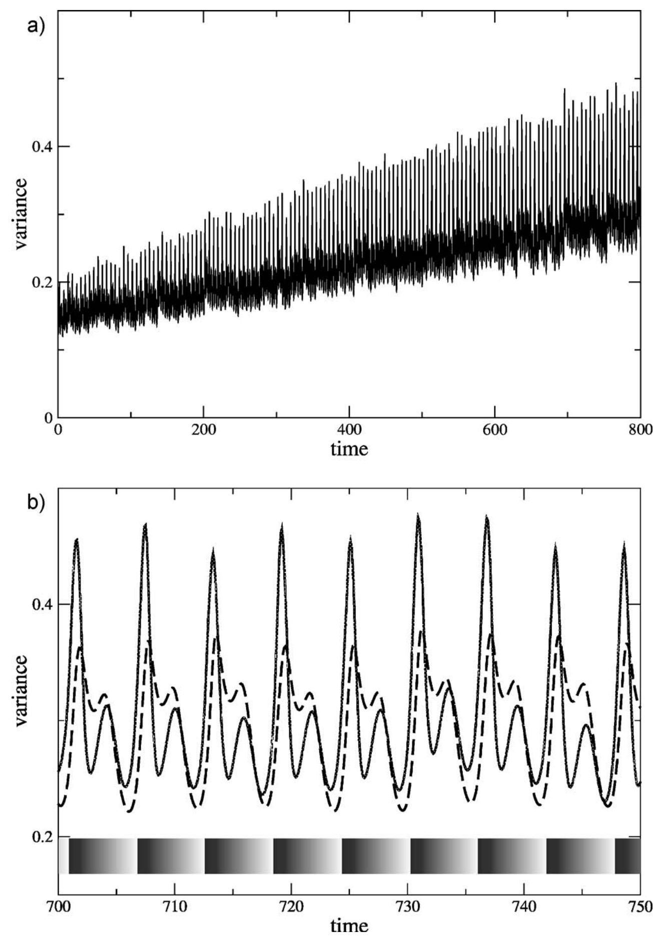

FIG. 3. (a) Variance evolution over a sample of 1000 independent Rössler systems. (b) Same as (a) within a time window of 700–750 (solid line). The dashed line indicates the modeled variance of the phase fluctuations. The mean phase of the oscillators is shown by the grey-scale strip on the lower part of the graph, ranging from 0 (black) to 2π (white).

$$\mathrm{Var}[\varphi(t)] \approx \mathrm{Var}[\varphi(0)] + D_{\langle A_t\rangle,\langle\varphi(t)\rangle}t, \tag{12}$$

where $\mathrm{Var}[\varphi(0)]\neq 0$, $\langle A_t\rangle$ is the mean amplitude and $\langle\varphi(t)\rangle$ the mean phase.

To check the validity of the model assumptions, the variance evolution over a sample of 1000 independent Rössler systems is simulated. The time step is chosen to be $\Delta t=0.1$. Figure 3(a) shows an increasing (in mean) variance of the phases, superposed by some spiking behavior.

The diffusion constants D_x, D_y in Eq. (12) are fitted to the simulations using a linear fit algorithm [16]. The identified parameters are $D_x=0.0089\pm 4\times 10^{-6}$, $D_y=0.0092\pm 6\times 10^{-6}$, thus different from zero. A comparison of the modeled variance, Eq. (11), with the simulation is shown in Fig. 3(b). The comparison shows that our model captures most of the structure but the modeled variance evolution seems to be low-pass filtered. This effect is probably due to the assumed diffusion constants in the *x*-*y* plain which are not depending on the state of the system. The constants D_x, D_y are therefore representing mean diffusion coefficients leading to a smoother curve for the variance evolution of the fluctuations.

Furthermore, the mean phase, grey-scale strip in Fig. 3, reveals that the burst of the z component is followed by spikes in the variance.

Beside the phase fluctuations emerging from the system's equations, a contribution of numerical noise is always present in the simulations. This noise corruption is contained in the identified coefficients D_x and D_y. The chosen integration accuracy $\epsilon = 10^{-12}$ gives a rough estimate on the numerical error of each time step Δt, see, e.g., Ref. [16]. Note that ϵ cannot be made arbitrarily small, because if ϵ is close to the machine precision the number of internal steps for integrating the whole time step Δt diverges. Since $\epsilon^2/\Delta t$ is several orders of magnitude smaller than D_x and D_y, the numerical error can be neglected in our analysis.

So far, we have derived an approximation of the phase dynamics by a diffusive process. It should now be verified if such phase diffusion satisfy the mixing condition of Eq. (1). Suppose that the diffusion is constant, such that the sampled phase evolution reads

$$\varphi_{k+1} = \varphi_k + \omega \Delta t + \sqrt{D\Delta t}\epsilon_k, \tag{13}$$

for some $D \neq 0$ and ϵ_k is again a sequence of uncorrelated white noise. Starting at φ_0 and taking the wrapped phase $\phi_k = \varphi_k \bmod 2\pi$ to gain a stationary process, the conditional probability density $\rho(\phi_\infty | \varphi_0) = \lim_{k\to\infty} \rho(\phi_k | \varphi_0)$ is thus

$$\rho(\phi_\infty|\varphi_0) = \lim_{k\to\infty} \frac{1}{\sqrt{2\pi D\Delta t k}} \times \sum_{j=-\infty}^{\infty} \exp\left(-\frac{(\omega\Delta t k + \varphi_0 - \phi_k - 2\pi j)^2}{2D\Delta t k}\right) = \frac{1}{(2\pi)^{3/2}} \int e^{-t^2/2} dt = \frac{1}{2\pi}. \tag{14}$$

Since $\rho(\phi_\infty | \varphi_0)$ does not depend on the initial value φ_0, the asymptotic independence of Eq. (1) is shown. Moreover, the same result holds for the phase difference of two independent processes, and therefore $R_{1,1} = 0$. If D is not constant with respect to sampling point of index k but greater than zero, the result of Eq. (14) does not change.

Extracting the mean diffusion constant of about $D_{\text{phase}} = 2.1 \times 10^{-4}$ from Fig. 3, the presence of the finite size effect for the synchronization analysis can be verified for the most simple model given in Eq. (13). In order to compare the outcome with the results presented in Sec. III, the parameters are chosen to be $\Delta t = 0.01$, $\omega = 1$, and $N = 131\,072$.

The distribution of the synchronization index $R_{1,1}$ is shown in Fig. 4. Since almost all mass is close to unity the finiteness of the amount of data has a predominant effect. Additionally, the value in case of the Rössler system $R_{1,1} = 0.92$ lies within the distribution but is slightly smaller than the mean synchronization index of the simplified model. This situation is exactly what one expects, because the bursts in the local diffusion rate destroy autocorrelations of the process. Due to Eq. (9) finite size effects are therefore slightly reduced. Finally, this positive result supports the strong presence of effects due to the finite amount of data.

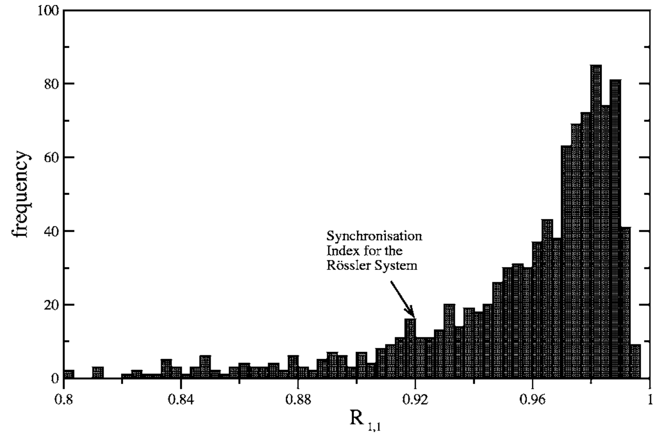

FIG. 4. Distribution of the synchronization index $R_{1,1}$ for two independent processes of type (13). The parameters D, Δt, ω and the amount of data N are chosen to allow a comparison of the results presented in Sec. III.

V. CONCLUSION

The discussion about the phase evolution for the Rössler system has a long history. Crutchfield *et al.* [24] claimed that the attractor topology is mainly responsible for the sharp peak, namely that trajectories are revolving a single hole. This conjecture cannot hold in general, because the peak of the Rössler system is much broader when, e.g., the parameters are chosen to be $a = b = 0.2$, $c = 13$. The attractor topology remains the same in this setting. An intermittent behavior of the phase was discussed but this hypothesis was rejected afterwards [25–27]. Recently, Anishchenko *et al.* determined an effective diffusion coefficient by fitting Lorentzian to peaks in the power spectrum [28–30]. The presented work therefore supports their hypothesis, that the chosen length of the time series is sufficiently large such that the spectral linewidth can be resolved.

We derived a model of the phase fluctuations of the Rössler system from the system's equations mainly under the assumptions of diffusion. Properties of this model are compared to simulated data. We have shown that the model captures the qualitative feature of the data. The diffusion constants derived from the model fitted to the data are significantly different from zero. In addition, a simplified but definitely mixing model of the phase evolution shows almost the same spurious synchronization index. This suggests that the Rössler system for the chosen set of parameters is mixing. However, the rate of mixing is extremely low, explaining the spurious results for the cross-spectral and the synchronization analysis as finite size effects.

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[1] P. Billingsley, *Probability and Measure* (Wiley, New York, 1995).

[2] O. Rössler, Phys. Lett. **57A**, 397 (1976).

[3] A. Lasota and M. Mackey, *Chaos, Fractals and Noise - Stochastic Aspects of Dynamics* (Springer, New York, 1994).

[4] D. Ruelle, *Thermodynamic Formalism* (Addison-Wesley, New York, 1978).

[5] D. Ruelle, Phys. Rev. Lett. **56**, 405 (1986).

[6] M. Pollicott, Invent. Math. **85**, 147 (1986).

[7] J. D. Farmer, Phys. Rev. Lett. **47**, 179 (1981).

[8] M. Khodas, S. Fishmann, and O. Agam, Phys. Rev. E **62**, 4769 (2000).

[9] S. Fishmann and S. Rahav, in *Lecture Notes in Physics*, edited by P. Garbaczewski and R. Olkiewicz (Springer, New York, 2002), pp. 165–192.

[10] P. Brockwell and R. Davis, *Time Series: Theory and Methods* (Springer, New York, 1987).

[11] E. Hannan, *Multiple Time Series* (Wiley, New York, 1970).

[12] M. Priestley, *Spectral Analysis and Time Series* (Academic, New York, 1989).

[13] J. Timmer, M. Lauk, W. Pfleger, and G. Deuschl, Biol. Cybern. **78**, 349 (1998).

[14] J. Timmer, M. Lauk, S. Häußler, V. Radt, B. Köster, B. Hellwig, B. Guschlbauer, C. Lücking, M. Eichler, and G. Deuschl, Int. J. Bifurcation Chaos Appl. Sci. Eng. **10**, 2595 (2000).

[15] D. Halliday, J. Rosenberg, A. Amjad, P. Breeze, B. Conway, and S. Farmer, Prog. Biophys. Mol. Biol. **64**, 237 (1995).

[16] W. Press, B. Flannery, S. Saul, and W. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, 1992).

[17] D. Gabor, J. Inst. Electr. Eng., Part 1 **93**, 429 (1946).

[18] B. Boashash, Proc. IEEE **80**, 519 (1992).

[19] A. Oppenheim and R. Schafer, *Digital Signal Processing* (Prentice-Hall, Englewood Cliffs, NJ, 1975).

[20] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[21] P. Tass, M. G. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkmann, A. Schnitzler, and H. J. Freund, Phys. Rev. Lett. **81**, 3291 (1998).

[22] F. Mormann, K. Lehnertz, P. David, and C. Elger, Physica D **144**, 358 (2000).

[23] T. Yalçinkaya and Y.-C. Lai, Phys. Rev. Lett. **79**, 3885 (1997).

[24] J. Crutchfield, J. Farmer, N. Packard, R. Shaw, G. Jones, and R. Donnelly, Phys. Lett. **76A**, 1 (1980).

[25] Y.-C. Lai, D. Armbruster, and E. Kostelich, Phys. Rev. E **62**, R29 (2000).

[26] A. Pikovsky and M. Rosenblum, Phys. Rev. E **64**, 058203 (2001).

[27] Y.-C. Lai, D. Armbruster, and E. J. Kostelich, Phys. Rev. E **64**, 058204 (2001).

[28] V. Anishchenko, T. Vadivasova, G. Okrokvertskhov, and G. Strelkova, Dyn. Chaos Radiophys. Electr. **48**, 824 (2003).

[29] V. S. Anishchenko, T. E. Vadivasova, J. Kurths, G. A. Okrokvertskhov, and G. I. Strelkova, Phys. Rev. E **69**, 036215 (2004).

[30] V. Anishchenko, T. Vadivasova, G. Okrokvertskhov, and G. Strelkova, Physica A **325**, 199 (2003).