Option Pricing in ARCH-type Models: with Detailed Proofs

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Abstract

ARCH-models have become popular for modelling financial time series. They seem, at first, however, to be incompatible with the option pricing approach of Black, Scholes, Merton et al., because they are discrete-time models and posess too much variability. We show that completeness of the market holds for a broad class of ARCH-type models defined in a suitable continuous-time fashion. As an example we focus on the GARCH(1,1)-M model and obtain, through our method, the same pricing formula as Duan (1995), who applied equilibrium-type arguments.

This is an extended version of Kallsen and Taqqu (1995). It includes additional comments and detailed proofs. It also includes a chapter concerning "the equality of filtrations" which deals with the following issue. Trading strategies should be based on information (filtration) that traders posess. In practice, however, one typically assumes that thay are predictable with respect to the filtration generated by a Brownian motion which serves as a background source of randomness. It is thus necessary to show that the two filtrations coincide. We do this here.

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Chapter 1

Pricing options in ARCH-like models

1.1 Introduction

Many papers have now been written on the pricing of stock options. One common approach – initiated by the work of Black, Scholes and Merton in the early seventies – is to assume that the underlying stock price behaves like a specific stochastic process and to make further assumptions about how trade takes place in order to finally derive an option pricing formula.

In the Black-Scholes model, for example, the underlying stock price follows geometric Brownian motion. In addition, the market is assumed to be frictionless, to allow for continuous trading and – this is important – to offer no arbitrage opportunities. These conditions are enough to ensure completeness of the market, that is to derive a fair price for various types of contingent claims. The property of completeness, however, often ceases to hold when geometric Brownian motion is replaced by some other process.

In order to determine option prices when there is no completeness further assumptions concerning risk premia and/or traders' preferences are usually made. This is the case for example in the models considered by Hull & White (1987), Johnson & Shanno (1987), Scott (1987), Wiggins (1987), Stein & Stein (1991), Melino & Turnbull (1990), Heston (1993) and Duan (1995).[1] It may very well be the case that many realistic models – those that meet statistical scrutiny – lack completeness.

However, one appealing feature of the Black-Scholes pricing formula is that it relies – apart from assumptions concerning the stock price behaviour – mainly on the fact that the market offers no "free lunches," i.e., no arbitrage. It seems thus worthwhile to investigate whether stock price models that are more realistic than geometric Brownian motion continue to fit into such a framework.

In the past dozen years ARCH-models have become popular for modelling financial time series since they are able to account for several empirical features like volatility clustering and leptokurtosis (fat tails) in the distribution of returns. While they differ substantially in their detailed expression, most ARCH-models involve a sequence of uncorrelated innovations whose

[1] Other approaches to option pricing under specific assumptions include Cox & Ross (1976), Merton (1976), Geske (1979) and Rubinstein (1983). For an overview see Hull (1993).

variance is random. Conditioned on the past, the variance depends only on the previous innovations and previous conditional variances. Typically, large (resp. small) absolute innovations increase (decrease) the conditional variance and therefore subsequent absolute innovations tend to be large (small) again. This leads to volatility clustering.

Viewed as discrete-time models, ARCH-models do not allow for option pricing along the lines of Black & Scholes (1973), Cox, Ross, Rubinstein (1979) and Harrison & Pliska (1981), because they are not complete.[2] Roughly speaking, there is too much variability in the stock price between successive time steps. Our way out, is to consider continuous-time ARCH-type models. These are models where the variance is a deterministic function of the past returns.

In Section 1.2 we investigate a general ARCH-like continuous time model and establish completeness. We show, in Section 1.3, how to extend the usual discrete-time ARCH models to continuous time so that they fit into this framework. This is done by letting the process evolve like a geometric Brownian motion between any two discrete ARCH times. This point of view is often taken implicitly in practice where, in order to estimate the volatility at a discrete ARCH time, one assumes that the volatility is constant between these times and uses high frequency return data to estimate it (see for example Taylor and Xu (1995) and Christensen and Prabhala (1994)).

We use the GARCH(1,1)-M model to illustrate our methodology. The pricing formula we obtain for the corresponding continuous time model coincides in this case with one considered by Duan (1995), who derived it based on the discrete-time model by using equilibrium-type arguments. In addition to obtaining pricing formulas and trading strategies (these have complicated expressions) we focus on the delta of the option, i.e., the first derivative of the option price with respect to the stock. If the underlying variables are not chosen in the right way then the delta of the option does not yield the correct strategies. We show how to select the right variables.

A brief conclusion is given in Section 1.4. The proofs can be found in Chapter 2. Finally, Chapter 3 deals with "the equality of filtrations" and contains the proof of Lemma 2 that is needed to establish completeness with respect to the appropriate information structure.

1.2 Completeness of a general ARCH-like model

We use the now classical mathematical setting of Harrison & Pliska (1981). Our general model for a market consisting of one kind of stock and bond is the following:

Prerequisites. Let $(\Omega, \mathcal{F}, P)$ be a probability space, T a positive real number (the terminal time), $(B_t)_{0\leq t\leq T}$ a standard Brownian motion on $(\Omega, \mathcal{F}, P)$, and $(\mathcal{F}_t)_{0\leq t\leq T}$ the P-completion of the filtration generated by $(B_t)_{0\leq t\leq T}$. We assume $\mathcal{F} = \mathcal{F}_T$. The filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ fulfills the usual conditions, i.e., $\mathcal{F}_0$ contains all null sets of P and $(\mathcal{F}_t)_{0\leq t\leq T}$ is right continuous (e.g. Protter (1990), Theorem I.4.31).

[2] One therefore has to make more assumptions in order to price contingent claims. In his paper about option pricing in ARCH-models, Duan (1995) makes assumptions concerning traders' utility functions. In their empirical study, Engle & Mustafa (1992) assume that the risk-neutral probability measure is that of an ARCH-model and estimate its parameters by equating theoretical and observed option prices.

The Brownian motion B will be the only source of randomness in the model considered below.

The model. As in Harrison & Pliska (1981), the *bond* is represented by a stochastic process $(S_t^0)_{0\le t\le T}$. For simplicity we assume a fixed constant interest rate r, i.e., S^0 is a function of the form $S_t^0 = e^{rt}$.

The *stock* in our model is the stochastic process $(S_t^1)_{0\le t\le T}$ meeting the following conditions:

$$S_t^1 = S_0^1 \exp\left[\int_0^t \left(\mu(\sigma_{s-}) - \frac{\sigma_{s-}^2}{2}\right) ds + X_t\right], \tag{1.2.1}$$

where the *drift* $\mu : \mathbf{R}^+ \to \mathbf{R}$ is a given function with continuous first derivative and where the process $(X_t)_{0\le t\le T}$ solves the stochastic differential equation

$$X_t = \int_0^t \sigma_{s-}\, dB_s \tag{1.2.2}$$

with

$$\sigma := F(X) > 0, \tag{1.2.3}$$

where $F : D[0,T] \to D[0,T]$ is a given functional Lipschitz operator. Observe that $\sigma_{t-} = F(X)_{t-}$ depends on $(X_s)_{0\le s<t}$, that is, on the past values of X. Assume, moreover, that F has a lower bound $K > 0$ and that $\limsup_{x\to\infty} \mu(x)/x < \infty$.

The motivation. To obtain some insight into the model apply the Itô-formula to (1.2.1) and observe that S solves the stochastic differential equation

$$S_t^1 = S_0^1 + \int_0^t \mu(\sigma_{s-}) S_s^1\, ds + \int_0^t \sigma_{s-} S_s^1\, dB_s, \tag{1.2.4}$$

or, in differential notation,

$$\frac{dS_t^1}{S_t^1} = \mu(\sigma_{t-})\, dt + \sigma_{t-}\, dB_t. \tag{1.2.5}$$

This means that the *relative stock price change* (or *instantaneous return*) has a drift component $\mu\, dt$ (maybe dependent on σ) and a noise component $\sigma_{t-}\, dB_t$. Equation (1.2.1) coincides for constant μ and σ with the stock price in the Black-Scholes model. Our model, however, allows for a changing volatility. More precisely, σ_{t-} is a function of X up to time t. Since X is – ignoring the drift μ – the integral of the instantaneous return in (1.2.5), equation (1.2.3) expresses the volatility in terms of past returns and is therefore an ARCH-type relationship.

Our setting is related to the models by Hull & White (1987), Scott (1987), Wiggins (1987), Stein & Stein (1991), and Heston (1993) in that S follows equation (1.2.1) with a time-varying volatility σ. The difference lies in the specification of σ. Here it is a function of past returns, whereas in these models, it involves a second source of randomness.

The following remarks clarify the assumptions in the model.

Remarks.

1. $D[0,T]$ denotes the space of functions $f : [0,T] \to \mathbf{R}$ that are *càdlàg*, i.e., right-continuous with existing left-hand limits. Almost all integrands in stochastic integrals occuring in this paper are *càglàd* (left-continuous with existing right-hand limits) so that integrability is ensured.

2. $F : D[0,T] \to D[0,T]$ is called *functional Lipschitz* if for all $f, g \in D[0,T]$ the following conditions hold:

   (i) For any t, $F(\cdot)_t : D[0,T] \to \mathbf{R}$ is measurable with respect to the σ-field of $D[0,T]$ that is generated by the projections $\alpha_s : D[0,T] \to \mathbf{R}$, $f \mapsto f(s)$, $s \le t$.

   (ii) for all t we have that $f|_{[0,t)} = g|_{[0,t)}$ implies $F(f)|_{[0,t)} = F(g)|_{[0,t)}$

   (iii) there exists a constant $K < \infty$ (independent of f, g) such that for all t we have $|F(f)_t - F(g)_t| \le K \sup_{s \le t} |f_s - g_s|$.

   Condition (i) implies that $F(X)$ is adapted for any adapted càdlàg process X. Therefore this definition is a special case of the one given in Protter (1990), V.3. or the one given in Chapter 3. Observe that $(\sigma_{t-})_{0 \le t \le T}$ can be used as an integrand, since σ is càdlàg.

3. According to Theorem 12 in Chapter 3 (or Protter (1990), Theorem V.3.7), there exists a unique solution to the stochastic differential equation (1.2.2). Therefore X and S^1 are uniquely defined by (1.2.1), (1.2.2), (1.2.3) (provided S^1_0, μ, F are given). Furthermore X and S^1 are continuous since they are stochastic integrals with respect to continuous processes.

4. The assumptions concerning the lower bound of F and the limiting behaviour of μ ensure that the process $((\mu(\sigma_t) - r)/\sigma_t)_{0 \le t \le T}$ is a bounded adapted càdlàg process.

We now introduce the *discounted price process* $(Z^0_t)_{0 \le t \le T}$:

$$Z_t := \frac{S^1_t}{S^0_t} = e^{-rt} S^1_t. \tag{1.2.6}$$

With the help of Girsanov's theorem we obtain

Lemma 1 *1. There is a well-defined probability measure P^* equivalent to P such that*

$$\frac{dP^*}{dP} := \exp\left[\int_0^T -\frac{\mu(\sigma_{s-}) - r}{\sigma_{s-}}\, dB_s - \frac{1}{2}\int_0^T \left(\frac{\mu(\sigma_{s-}) - r}{\sigma_{s-}}\right)^2 ds\right]$$

is its Radon-Nikodym density.

2. The discounted price process Z is a positive local martingale (and also a supermartingale) with respect to P^*, and is given by

$$Z_t = Z_0 \exp\left\{-\int_0^t \frac{\sigma^2_{s-}}{2}\, ds + \int_0^t \sigma_{s-}\, dW_s\right\} \tag{1.2.7}$$

where $(W_t)_{0\leq t\leq T}$*, defined by*

$$W_t := \int_0^t \frac{\mu(\sigma_{s-}) - r}{\sigma_{s-}}\, ds + B_t, \tag{1.2.8}$$

is a standard Brownian motion with respect to P^**.*

In the beginning of this chapter we assumed that the filtration under consideration is the filtration generated by B. From an intuitive viewpoint however, the filtration generated by S (or Z) rather than B is the more natural one, since S is the actually observed process. The following lemma, proved in Section 3.4, shows that we need not worry about this point in our setting. (See also the remark following Theorem 3.)

Lemma 2 *The* P*-completed filtrations generated by either* B*,* W*,* S*, or* Z *coincide with* $(\mathcal{F}_t)_{0\leq t\leq T}$*.*

We now recall some definitions and statements from Harrison & Pliska (1981). Suppose that Z is a martingale under P^*. A process $\phi = (\phi_t^0, \phi_t^1)_{0\leq t\leq T}$ is called *trading strategy* if ϕ is predictable and

$$\left(\int_0^t (\phi_s^1)^2\, d[Z,Z]_s\right)^{\frac{1}{2}}, \ 0 \leq t \leq T, \text{ is locally integrable under } P^*. \tag{1.2.9}$$

We say that a trading strategy ϕ is *admissable* if it is *self-financing*, i.e., the *discounted value process* $V(\phi) := \phi^0 + \phi^1 Z$ solves

$$V(\phi) = V_0(\phi) + \int_0^\bullet \phi^1 dZ, \tag{1.2.10}$$

and if, in addition, $V(\phi)$ is a non-negative martingale under P^* (V is the V^* of Harrison & Pliska (1981)). A *contingent claim* is a positive random variable C. We call it *attainable* if there exists an admissable strategy ϕ that *generates* C, i.e., $V_T(\phi) = e^{-rT}C$. For such a claim C, $\pi_0 := V_0(\phi) = E_{P^*}(e^{-rT}C)$ is called the *price associated with* C and this is the only reasonable price for C at time 0 if we assume the absence of arbitrage opportunities. For times t between 0 and T the fair price of the claim is given by $\pi_t = e^{rt}V_t(\phi) = e^{rt}E_{P^*}(e^{-rT}C|\mathcal{F}_t)$. We call a market *complete* if every P^*-integrable claim is attainable. A P^*-martingale Y is said to have the *representation property* if any martingale M with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P^*)$ can be written as $M = M_0 + \int H dY$ for some $H \in \mathcal{L}(Y)$, where $\mathcal{L}(Y)$ denotes the set of all predictable processes such that $(\int_0^t (H_s)^2 d[Y]_s)^{\frac{1}{2}}$, $0 \leq t \leq T$ is locally integrable under P^*. If the discounted price process Z has the representation property, then the market is complete in the above sense.

Theorem 3 *Suppose that* Z *is a martingale under* P^**. Then the model is complete. In particular,* $\pi_0 = E_{P^*}(e^{-rT}C)$ *is the price at time* 0 *for a given integrable contingent claim* C *(e.g., the European call option with expiration date* T *and strike price* K *defined by* $C = (S_T^1 - K)^+$*).*

Remarks.

1. In order to derive fair option prices Z must be a martingale, not just a local martingale. This is why we assume in Theorem 3 that Z is a martingale under P^*. Since Z is a positive supermartingale (Lemma 1), in order to show that Z is a martingale under P^* it is enough to prove that $E(Z_T) = Z_0$, where T is the terminal time (see Harrison & Pliska (1981), 3.9).

2. The next remark underlines the importance of filtrations in contingent claim pricing.

   The proof of Theorem 3 makes use of Lemma 2. We derive the representation property of Z with respect to $(\mathcal{F}_t)_{0\le t\le T}$ from the representation property of W with respect to its own natural filtration, which was possible because the filtrations coincide. One can still prove Theorem 3 without knowing that the filtrations generated by B, W and Z tally by using a result that can be found e.g. in Stroock & Yor (1980), Lemma 8.1. Applied to our situation it yields that since $(\mathcal{F}_t)_{0\le t\le T}$ is the natural filtration of the Brownian motion B and, since W and B are related to each other by Girsanov's theorem (see the proof of Lemma 1), W has the predictable representation property with respect to $(\mathcal{F}_t)_{0\le t\le T}$ and the probability measure P^*.

   However, Theorem 3 would be unsatisfactory from an intuitive point of view if one ignored the equality of filtrations given in Lemma 2. Indeed, completeness of the model implies the existence of a unique fair option price π_0 which is also achievable by using trading strategies. These trading strategies, however, should be based on information that traders posess. This will not be the case if we only claim that trading strategies are predictable with respect to $(\mathcal{F}_t)_{0\le t\le T}$ but not with respect to the natural filtration of S (or Z) which represents the information that is really available to traders. Lemma 2 removes this difficulty.

1.3 Example: GARCH(1,1)-M

Our interest lies – as indicated in Section 1.1 – in models that can be made complete so that options can be priced assuming only the absence of arbitrage. For a discrete-time market consisting of only two securities "stock" and "bond" to be complete it is necessary, roughly speaking, that over any single time period the stock price has at most only two possible values to move to (see Harrison & Pliska (1981), Willinger & Taqqu (1987)). The stock may go up or down but it is not supposed to take several or worse, infinitely many values, as in the case of ARCH.

We want to indicate a way to sidestep this difficulty. The idea is to interpolate the usual (e.g. daily) ARCH models in a continuous-time fashion. As an example we consider the particular ARCH-model known as GARCH(p,q)-M.

Denoting by S_t^1 the stock price at time t we may formulate the GARCH(p,q)-M model as follows:

$$\log \frac{S_t^1}{S_{t-1}^1} = \mu(\sigma_t) - \frac{\sigma_t^2}{2} + \sigma_t \epsilon_t, \tag{1.3.1}$$

where μ is a given function, $\epsilon_1, \epsilon_2, \ldots$ is a sequence of i.i.d. standard normal random variables, and σ_t satisfies:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i (\sigma_{t-i}\epsilon_{t-i})^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{1.3.2}$$

$\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ being fixed constants.

The innovations in this model are $x_t := \sigma_t \epsilon_t$, $t = 1, 2, \ldots$. Their variance (i.e., σ_t^2) conditioned on "the past" is given by the GARCH(p,q)-equation (1.3.2), which indicates that σ_t^2 is a linear function of the earlier squared innovations $x_{t-1}^2, x_{t-2}^2, \ldots, x_{t-p}^2$ and the conditional variances $\sigma_{t-1}^2, \sigma_{t-2}^2, \ldots, \sigma_{t-q}^2$. The drift in the return in equation (1.3.1), namely $\mu(\sigma_t) - \sigma_t^2/2$, also depends on σ_t. Models with this property are known as ARCH-in-mean or ARCH-M. The meaning of equation (1.3.2) is that high volatility can result from large absolute returns x_{t-i}^2 or from a large volatility σ_{t-j}^2 in the preceding time periods. For more details on ARCH-models, see Bollerslev et al. (1992).

1.3.1 Continuous-time GARCH

We now illustrate the continuous time embedding methodology with the GARCH(1,1)-M model. We will replace the i.i.d. random variables ϵ_t in (1.3.1) and (1.3.2) by increments $B_{[t]} - B_{[t]-1}$ of a standard Brownian motion B. More specifically we assume the following continuous time model:

Let $\sigma_0, \omega, \alpha, \beta$ be positive real numbers and let $\mu : \mathbf{R}^+ \to \mathbf{R}$ be a given function with continuous first derivative and such that $\limsup_{x\to\infty} \mu(x)/x < \infty$. Now define S_t, X_t, σ_t, Z_t as in Section 1.2 with

$$F(X)_t = \begin{cases} \sigma_0 & \text{for } 0 \le t < 1 \\ \left(\omega + \alpha(X_{[t]} - X_{[t]-1})^2 + \beta F(X)_{[t]-1}^2\right)^{\frac{1}{2}} & \text{for } t \ge 1. \end{cases} \tag{1.3.3}$$

Explicitly:

$$S_t^1 = S_0^1 \exp\left[\int_0^t \left(\mu(\sigma_{s-}) - \frac{\sigma_{s-}^2}{2}\right) ds + X_t\right], \tag{1.3.4}$$

$$X_t = \int_0^t \sigma_{s-}\, dB_s, \tag{1.3.5}$$

$$\sigma := F(X), \tag{1.3.6}$$

$$Z_t := e^{-rt} S_t^1. \tag{1.3.7}$$

Note that $F \ge \sqrt{\omega} > 0$. The processes S, X, σ, Z are well defined because

Lemma 4 *F is functional Lipschitz.*

In order to see that this model is in fact an extension of (1.3.1), (1.3.2) observe that for integer values of t,

$$\log \frac{S_t^1}{S_{t-1}^1} = \mu(\sigma_{t-}) - \frac{\sigma_{t-}^2}{2} + \sigma_{t-}(B_t - B_{t-1})$$

and

$$\sigma_{t-}^2 = (F(X)_{t-})^2 = \begin{cases} \sigma_0^2 & \text{for } t = 1 \\ \omega + \alpha\sigma_{(t-1)-}^2(B_{t-1} - B_{t-2})^2 + \beta\sigma_{(t-1)-}^2 & \text{for } t \geq 2. \end{cases}$$

Thus for $n \leq t < n+1$, the price process S_t behaves like geometric Brownian motion with drift $\mu(\sigma_n)$ and volatility σ_n. The parameter σ_n is in fact random and changes from one integer time to the next according to the GARCH(1,1)-M model.

1.3.2 Completeness

Assuming again a constant interest rate r, we note that Lemmas 1 and 2 from Section 1.2 apply. Here the equivalent measure P^* is given by

$$\begin{aligned} \frac{dP^*}{dP} &= \exp\left\{ \int_0^t -\frac{\mu(\sigma_{s-}) - r}{\sigma_{s-}}\, dB_s - \frac{1}{2}\int_0^t \left(\frac{\mu(\sigma_{s-}) - r}{\sigma_{s-}}\right)^2 ds \right\} \\ &= \exp\left\{ \sum_{i=1}^{[T]} \left[\left(-\frac{\mu(\sigma_{i-1}) - r}{\sigma_{i-1}}\right)(B_i - B_{i-1}) - \frac{1}{2}\left(\frac{\mu(\sigma_{i-1}) - r}{\sigma_{i-1}}\right)^2 \right] \right. \\ &\qquad \left. + \left(-\frac{\mu(\sigma_{[T]}) - r}{\sigma_{[T]}}\right)(B_T - B_{[T]}) - \frac{1}{2}\left(\frac{\mu(\sigma_{[T]}) - r}{\sigma_{[T]}}\right)^2 \right\}. \end{aligned}$$

Lemma 5 *Z is a martingale under P^*.*

Combining this result with Theorem 3 we obtain

Theorem 6 (Completeness). *Our continuous-time GARCH(1,1)-M model is complete and the price for any integrable claim C is*

$$\pi_0 = E_{P^*}(e^{-rT}C). \tag{1.3.8}$$

Remarks.

1. It is not necessary to modify the model if the time t for which we want the option price does not coincide with the beginning of an ARCH time step (e.g. a day). The price of the option for any time $0 \leq t \leq T$ is

$$\pi_t = e^{rt}E_{P^*}(e^{-rT}C|\mathcal{F}_t). \tag{1.3.9}$$

2. The derivation of the option price works in the same way for other ARCH-models, in particular for GARCH(p, q)-M, where we consider

$$F(X)_t = \sigma_{[t]} \quad \text{for } 0 \leq t < \max(p,q)$$

and

$$F(X)_t = \left(\omega + \sum_{i=1}^{p}\alpha_i(X_{[t]-(i-1)} - X_{[t]-i})^2 + \sum_{j=1}^{q}\beta_j F(X)_{[t]-j}^2\right)^{\frac{1}{2}} \quad \text{for } t \geq \max(p,q).$$

(with fixed constants $\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \sigma_0, \ldots, \sigma_{\max(p,q)-1}$) instead of equation (1.3.3).

Let us now look at how the discounted stock price Z behaves under the equivalent martingale measure P^*. Using relations (1.2.8) and (1.2.6), we can replace in equations (1.3.3) – (1.3.6) the process B (the standard Brownian motion with respect to P) by W (the standard Brownian motion with respect to P^*) and also the stock price S by the discounted price process Z and get

$$Z_t = Z_0 \exp\left[-\int_0^t \frac{\sigma_{s-}^2}{2}\, ds + Y_t\right] \tag{1.3.10}$$

with

$$Y_t = \int_0^t \sigma_{s-}\, dW_s \tag{1.3.11}$$

and

$$\sigma_t^2 = \begin{cases} \sigma_0^2 & \text{for } 0 \le t < 1 \\ \omega + \alpha\left(Y_{[t]} - Y_{[t]-1} + \mu(\sigma_{[t]-1}) - r\right)^2 + \beta\sigma_{[t]-1}^2 & \text{for } t \ge 1. \end{cases} \tag{1.3.12}$$

We may now observe an interesting difference with the Black & Scholes case. The discounted price Z_t in (1.3.10) is a function of the volatility σ but because the evaluation of σ, as given by (1.3.12), involves μ, the option price at time 0, namely $\pi_0 = E_{P^*}(e^{-rT}C)$, is a function of the drift μ of the stock. This will be the case even if we choose μ constant. This, in fact, is the typical state of affairs. The Black & Scholes case is, in this regard, a degenerate situation.

Equations (1.3.10) – (1.3.12) and Theorem 6 correspond to Theorem 2.2 and Corollary 2.3 in Duan (1995). Therefore Duan's option pricing formula coincides for integer times with ours, if we choose the same type of drift function as Duan ($\mu(\sigma_t) = r + \lambda\sigma_t$ for some positive constant λ).

1.3.3 Pricing formula and trading strategies

We want to derive a more detailed expression of the price π_t given in (1.3.9). To this end we consider a claim of the form $C = \psi(Z_T)$ for some measurable, $P^*_{Z_T}$-integrable function $\psi : \mathbf{R} \to \mathbf{R}^+$. In the following, fix an integer $n < T$ and let $n \le t < \min(n+1, T)$.

Theorem 7 (Pricing formula). *The discounted price of the claim $C = \psi(Z_T)$ at time t equals*

$$\begin{aligned} e^{-rt}\pi_t &= p_n(Z_t, Z_n, \sigma_n, t, T) \\ &:= \int \psi\Big\{ Z_t \exp\Big[-\frac{\sigma_n^2}{2}(n+1-t) + \sigma_n x_1 \\ &\quad + \sum_{i=1}^{[T]-n} \frac{1}{2} f^i\Big(\sigma_n^2, x_1 + \frac{1}{\sigma_n}\log\frac{Z_t}{Z_n} - \frac{\sigma_n}{2}(t-n), x_2, \ldots, x_i\Big) \\ &\quad - \frac{1}{2} f^{[T]-n}\Big(\sigma_n^2, x_1 + \frac{1}{\sigma_n}\log\frac{Z_t}{Z_n} - \frac{\sigma_n}{2}(t-n), x_2, \ldots, x_{[T]-n}\Big)([T]+1-T) \\ &\quad + \sum_{i=1}^{[T]-n} \Big[f^i\Big(\sigma_n^2, x_1 + \frac{1}{\sigma_n}\log\frac{Z_t}{Z_n} - \frac{\sigma_n}{2}(t-n), x_2, \ldots, x_i\Big)\Big]^{\frac{1}{2}} x_{i+1}\Big]\Big\} \\ &\quad N(\mathbf{0}, diag(n+1-t, 1, \ldots, 1, T-[T]))d(x_1, \ldots, x_{[T]-n+1}) \end{aligned} \tag{1.3.13}$$

(resp. $N(0, T-t)(dx_1)$ for $[T] = [t]$),

where $n = [t]$ *and the functions* $f^k : \mathbf{R}^+ \times \mathbf{R}^k \to \mathbf{R}^+$, $k = 0, 1, 2, \ldots$, *are defined recursively by*

$$\begin{aligned} f^0(\sigma^2) &:= \sigma^2, \\ f^1(\sigma^2, x_1) &:= \omega + \alpha(\sigma x_1 + \mu(\sigma) - r)^2 + \beta\sigma^2, \\ f^{k+1}(\sigma^2, x_1, \ldots, x_{k+1}) &:= f^1(f^k(\sigma^2, x_1, \ldots, x_k), x_{k+1}). \end{aligned}$$

In order to obtain the generating trading strategies we need the following technical lemma which will allow us to apply Itô's formula. Recall that P^* is a probability measure on Ω and that the claim C is assumed integrable. We use the notation P^*_X to denote the probability measure induced by a random variable X.

Lemma 8 *Fix an integer* $n < T$. *For* $P^*_{(Z_n,\sigma_n)}$*-almost all* $(\hat{z}, \hat{\sigma})$ *the function* $(z,t) \mapsto p_n(z, \hat{z}, \hat{\sigma}, t, T)$ *has continuous second order partial derivatives in* $\mathbf{R}^+ \times (n, \min(n+1, T))$.

Theorem 9 (Trading strategies). *The following trading strategy* $\phi = (\phi^0, \phi^1)$ *defined by*

$$\phi^1_t = \begin{cases} D_1 p_n(Z_t, Z_n, \sigma_n, t, T) & \text{for } n < t < \min(n+1, T) \\ \left. \begin{cases} \lim_{s \uparrow t} \phi^1_s & \text{if the limit exists} \\ 0 & \text{else} \end{cases} \right\} & \text{for } t \text{ integer} \end{cases}$$

and

$$\phi^0_t = e^{-rt}\pi_t - Z_t \phi^1_t \quad \text{for } 0 \le t \le T.$$

generates the claim C. *(*D_1 *denotes the partial derivative with respect to the first argument, and* π_t *is given by (1.3.13)).*

Remarks.

1. Note that the definition of ϕ^1 for integer times is of no importance (as long as ϕ is predictable) since it does not affect the stochastic integrals. Note also that ϕ will usually have jumps at integer times (in the sense that $\lim_{t \uparrow\uparrow n} \phi^1_t \neq \lim_{t \downarrow\downarrow n} \phi^1_t$), since σ is typically discontinuous as well.

2. For higher order GARCH we get essentially the same result for the trading strategy. More specifically, for GARCH(p,q)-M we may express the discounted price at time $t \ge \max(p,q)$ as

$$e^{-rt}\pi_t = p_n(Z_t, Z_n, \ldots, Z_{n+1-p}, \sigma_n, \ldots, \sigma_{n+1-q}, t, T),$$

where p_n is again a measurable function that has continuous second order partial derivatives with respect to (Z_t, t). The statement and the proof of Theorem 9 apply analogously in this case.

3. In undiscounted terms we may write

$$\begin{aligned} \pi_t &= e^{rt} p_n(e^{-rt} S^1_t, e^{-rn} S^1_n, \sigma_n, t, T) \\ &=: \tilde{p}_n(S^1_t, S^1_n, \sigma_n, t, T) \end{aligned}$$

and thus have

$$\phi^1_t = D_1 \tilde{p}_n(S^1_t, S^1_n, \sigma_n, t, T) \quad \text{for } n < t < n+1.$$

4. Theorem 9 expresses the generating trading strategy in terms of the functional dependence of π_t on Z_t, Z_n, σ_n, t, T. As in the Black-Scholes case, the number of shares of stock in the duplicating portfolio is the partial derivative of the (discounted) option price with respect to the (discounted) stock price. It is often called the *delta of the option* in the economics literature and it also determines the optimal hedge (see e.g. Hull (1993), Section 13.5, Eades (1992), p. 217ff). Here, however, the function that has to be differentiated is not the same as in the Black-Scholes case; it depends on the additional variables $Z_{[t]}$ and $\sigma_{[t]}$.

   Note that the choice of the variables in the functional representation of π_t is of utmost importance. We could also have expressed π_t in terms of $(S_t^1, W_t - W_n, \sigma_n, t, T)$, say,

$$\pi_t = \hat{p}_n(S_t^1, W_t - W_n, \sigma_n, t, T). \tag{1.3.14}$$

   But the partial derivative $D_1\hat{p}_n(S_t^1, W_t - W_n, \sigma_n, t, T)$ does not, in general, equal $D_1\widetilde{p}_n(S_t^1, S_n^1, \sigma_n, t, T)$. Indeed, by (1.2.7), we have

$$\sigma_n(W_t - W_n) = (\frac{\sigma_n^2}{2} - r)(t - n) + \log S_t^1 - \log S_n^1 \tag{1.3.15}$$

   for $n \leq t < n + 1$, and thus the expression relating $W_t - W_n$ to S_n^1 involves S_t^1 as well. Hence $D_1\hat{p}_n(S_t^1, W_t - W_n, \sigma_n, t, T)$ does not correspond to the generating trading strategy.

1.3.4 Monte-Carlo simulation

Due to the complexity of the formulas it is not easy to find an explicit analytical expression for π and even numerical approximation seems to be a hard task. However, some Monte-Carlo simulations have been performed by Duan (1995), which give some insight in the features of this option pricing model.

As noted above, Duan's simulations (apart from those concerning the option delta) apply to our case. They reveal that if we are in a GARCH setting and wrongly use the Black-Scholes formula for option valuation, then inconsistencies similar to those observed in real markets appear, e.g., U-shaped implied volatilities ("smile") and underpricing of out-of-the-money options. In particular, it is generally not true that the option price at time t can be computed by plugging a good estimate for σ into the Black-Scholes formula. If this were true then we would still have implied volatilities independent of the strike price, i.e., no smile.

1.4 Conclusion

We have seen that it is possible to consider a general continuous-time ARCH-type setting which allows for option pricing based on absence of arbitage. We have analyzed the GARCH(1,1)-model in detail, but other commonly used ARCH-models fit into this framework as well. As in the Black-Scholes case, the delta of the option leads to a generating trading strategy; the pricing formula is more complicated and not given in a simple analytical form.

The procedure we demonstrated in Section 1.3 for GARCH(1,1)-M yields continuous-time models that still carry a strong discrete-time flavour. Namely their instantaneous variance changes only at the end of fixed time intervals (e.g. days). Future research will focus on models obtained by using smoother operators.

Chapter 2

Proofs

This chapter contains the proofs of the lemmas and theorems of Chapter 1. The proof of Lemma 2, however, is to be found in Chapter 3.

Proof of Lemma 1. Since $((\mu(\sigma_t) - r)/\sigma_t)_{0\le t\le T}$ is uniformly bounded, Girsanov's theorem (Protter (1990), Theorem III.6.21) yields the first statement of the lemma and shows that W is a standard Brownian motion under P^*, in particular a square integrable martingale. Moreover,

$$\begin{aligned} Z_t &= e^{-rt} S_t^1 \\ &= S_0^1 \exp\left\{\int_0^t \left(\mu(\sigma_{s-}) - r - \frac{\sigma_{s-}^2}{2}\right) ds + \int_0^t \sigma_{s-}\, dB_s\right\} \\ &= Z_0 \exp\left\{-\int_0^t \frac{\sigma_{s-}^2}{2}\, ds + \int_0^t \sigma_{s-}\, dW_s\right\} \qquad (2.0.1) \\ &= Z_0\, \mathcal{E}\left(\int_0^t \sigma_{s-}\, dW_s\right) \end{aligned}$$

($[\cdot,\cdot]$ denotes the quadratic variation and $\mathcal{E}$ the stochastic exponential). Therefore Z is a solution of

$$Z_t = Z_0 + \int_0^t Z_s \sigma_{s-}\, dW_s$$

(Protter (1990), Theorem II.8.36). Since $(Z_t\sigma_{t-})_{0\le t\le T}$ is an adapted left-continuous process and W is a locally square integrable martingale under P^*, we conclude that Z is a locally square integrable martingale as well (Protter (1990), Theorem II.5.20). Z is also a P^*-supermartingale, because any positive local martingale is a supermartingale (see e.g. Harrison & Pliska (1981), 3.8). □

Proof of Lemma 2. See Theorem 13 in Chapter 3. □

Proof of Theorem 3. The proof follows the lines of the proof for geometric Brownian motion in Harrison & Pliska (1981). Let M be a martingale with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P^*)$. Since W is Brownian motion under P^* and since, by Lemma 2, $(\mathcal{F}_t)_{0\le t\le T}$ is the P-completed filtration generated by W, we have

$$M_t = M_0 + \int_0^t \theta_s \, dW_s, \quad 0 \leq t \leq T,$$

for some predictable process $(\theta_t)_{0 \leq t \leq T}$ with $P^*(\int_0^T |\theta_t|^2 dt < \infty) = 1$ (e.g. Protter (1990), Theorem IV.3.42 and Corollary 2). The process $H_t := \theta_t/(\sigma_{t-} Z_t)$, $0 \leq t \leq T$, is well-defined because $\sigma_{t-} Z_t > 0$ for all t.

Since Z is continuous, $(H_t)_{0 \leq t \leq T}$ is predictable and

$$\left(\int_0^t (H_s)^2 \, d[Z,Z]_s \right)^{\frac{1}{2}} = \left(\int_0^t (H_s)^2 \sigma_{s-}^2 Z_s^2 \, ds \right)^{\frac{1}{2}} = \left(\int_0^t \theta_s^2 ds \right)^{\frac{1}{2}},$$

which is continuous, hence locally integrable. Moreover,

$$\int_0^t H_s \, dZ_s = \int_0^t \frac{\theta_s}{\sigma_{s-} Z_s} \, dZ_s = \int_0^t \frac{\theta_s}{\sigma_{s-} Z_s} \sigma_{s-} Z_s \, dW_s = \int_0^t \theta_s \, dW_s = M_t - M_0.$$

This proves, since M is arbitrary, that Z has the representation property (with respect to $(\mathcal{F}_t)_{0 \leq t \leq T}$, which is also the P^*-completed filtration generated by Z).

The integrability of the European call option follows from $0 \leq (S_T^1 - K)^+ \leq e^{rT} Z_T$ and $E(Z_T) < \infty$ (Lemma 1). □

Proof of Lemma 4. Observe first that $F : D[0,T] \to D[0,T]$ satisfies (i) and (ii) in the definiton of "functional Lipschitz". Fix $f, g \in D[0,T]$, $t \in [0,T]$. We prove (iii) by induction. Define $h(x,y) = \sqrt{\omega + \alpha x^2 + \beta y^2}$ for $x, y \in \mathbf{R}$. Letting $D := (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, we have that

$$Dh(x,y) = \left(\frac{\alpha x}{\sqrt{\omega + \alpha x^2 + \beta y^2}}, \frac{\beta y}{\sqrt{\omega + \alpha x^2 + \beta y^2}} \right)$$

and hence

$$\sup_{x,y \in \mathbf{R}^2} \|Dh(x,y)\| \leq \sqrt{\alpha} + \sqrt{\beta} =: M,$$

where here $\|(x,y)\| := |x| + |y|$ denotes the sum norm on $\mathbf{R}^2$. We prove by induction on n that $|F(f)_t - F(g)_t| \leq 2nM \sup_{s \leq t} |f_s - g_s|$ for all $t < n$. This relation holds for $n = 1$ because the left-hand side is 0. Assume it holds for n. Applying the mean value theorem on h, we obtain

$$\begin{aligned}
|F(f)_t - F(g)_t| &= |h(f_{[t]} - f_{[t]-1}, F(f)_{[t]-1}) - h(g_{[t]} - g_{[t]-1}, F(g)_{[t]-1})| \\
&\leq \sup_{x,y \in \mathbf{R}^2} \|Dh(x,y)\| \Big(|(f_{[t]} - f_{[t]-1}) - (g_{[t]} - g_{[t]-1})| \\
&\quad + |(F(f)_{[t]-1} - F(g)_{[t]-1})| \Big) \\
&\leq 2M \sup_{s \leq t} |f_s - g_s| + 2nM \sup_{s \leq t} |f_s - g_s| \\
&= 2(n+1) M \sup_{s \leq t} |f_s - g_s|.
\end{aligned}$$

Thus (iii) holds with $K := 2(T+1)M$. □

Proof of Lemma 5. We prove that $E_{P^*}(Z_t|\mathcal{F}_s) = Z_s$ for $[s] \leq s \leq t \leq [s]+1$. By induction over $[t]-[s]$ it follows $E_{P^*}(Z_t|\mathcal{F}_s) = Z_s$ for all $s \leq t$.

Let $[s] \leq s \leq t \leq [s]+1$ and let W be defined as in Equation (1.2.8). By (1.2.7)

$$\begin{aligned} E_{P^*}(Z_t|\mathcal{F}_s) &= E_{P^*}\left(Z_s \exp\left\{\int_s^t \sigma_{u-}\, dW_u - \int_s^t \frac{\sigma_{u-}^2}{2}\, du\right\}\middle|\mathcal{F}_s\right) \\ &= E_{P^*}\Big(Z_s \exp\Big\{\sigma_{[s]}(W_t - W_s) - \frac{\sigma_{[s]}^2}{2}(t-s)\Big\}\Big|\mathcal{F}_s\Big) \\ &= Z_s \exp\Big(-\frac{\sigma_{[s]}^2}{2}(t-s)\Big) E_{P^*}(\exp(\sigma_{[s]}(W_t - W_s)|\mathcal{F}_s). \end{aligned}$$

Since $W_t - W_s$ is independent of $\mathcal{F}_s$ under P^* and since $E(e^U) = e^{\frac{\sigma^2}{2}}$ $N(0,\sigma^2)$-distributed, we conclude that

$$E_{P^*}(\exp(\sigma_{[s]}(W_t - W_s))|\mathcal{F}_s) = \exp\Big(-\frac{\sigma_{[s]}^2}{2}(t-s)\Big)$$

and therefore $E_{P^*}(Z_t|\mathcal{F}_s) = Z_s$. □

Proof of Theorem 7. We have, by (1.2.7),

$$\begin{aligned} Z_T &= Z_t \exp\left[-\int_t^T \frac{\sigma_{s-}^2}{2}\, ds + \int_t^T \sigma_{s-}\, dW_s\right] \\ &= Z_t \exp\left[-\frac{\sigma_n^2}{2}(n+1-t) + \sum_{i=n+1}^{[T]} \frac{\sigma_i^2}{2} - \frac{\sigma_{[T]}^2}{2}([T]+1-T)\right. \\ &\qquad \left. + \sigma_n(W_{n+1} - W_t) + \sum_{i=n+1}^{[T]} \sigma_i(W_{i+1} - W_i) - \sigma_{[T]}(W_{[T]+1} - W_T)\right]. \end{aligned} \tag{2.0.2}$$

(Note that the undefined variable $W_{[T]+1}$ that has been introduced to simplify the computations appears twice and therefore cancels out.)

The expression for f^1 is based on (1.3.12): f^1 yields the σ^2 for the subsequent time period. Because of relations (1.3.11) and (1.3.12), we can rewrite (2.0.2) as

$$\begin{aligned} Z_T = Z_t \exp\Big[& -\frac{\sigma_n^2}{2}(n+1-t) + \sigma_n(W_{n+1} - W_t) \\ & + \sum_{i=n+1}^{[T]} \frac{1}{2} f^{i-n}(\sigma_n^2, W_{n+1} - W_n, \ldots, W_i - W_{i-1}) \\ & - \frac{1}{2} f^{[T]-n}(\sigma_n^2, W_{n+1} - W_n, \ldots, W_{[T]} - W_{[T]-1})([T]+1-T) \\ & + \sum_{i=n+1}^{[T]} \Big(f^{i-n}(\sigma_n^2, W_{n+1} - W_n, \ldots, W_i - W_{i-1})\Big)^{\frac{1}{2}} (W_{i+1} - W_i) \\ & - \Big(f^{[T]-n}(\sigma_n^2, W_{n+1} - W_n, \ldots, W_{[T]} - W_{[T]-1})\Big)^{\frac{1}{2}} (W_{[T]+1} - W_T)\Big]. \end{aligned}$$

Since $n \leq t < n+1$, (1.3.10) and (1.3.11) yield

$$\begin{aligned} \log \frac{Z_t}{Z_n} &= -\int_n^t \frac{\sigma_{s-}^2}{2}\, ds + \int_n^t \sigma_{s-}\, dW_s \\ &= -\frac{\sigma_n^2}{2}(t-n) + \sigma_n (W_t - W_n). \end{aligned} \tag{2.0.3}$$

Hence defining $g : \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+ \times [0,1] \to \mathbf{R}^+$ by

$$g(Z_t, Z_n, \sigma_n, t-n) := \frac{1}{\sigma_n} \log \frac{Z_t}{Z_n} - \frac{\sigma_n}{2}(t-n),$$

we may write $W_t - W_n = g(Z_t, Z_n, \sigma_n, t-n)$ and $W_{n+1} - W_n = W_{n+1} - W_t + W_t - W_n = W_{n+1} - W_t + g(Z_t, Z_n, \sigma_n, t-n)$. We then obtain the following expression for Z_T:

$$\begin{aligned} Z_T &= Z_t \exp\Big[-\frac{\sigma_n^2}{2}(n+1-t) + \sigma_n (W_{n+1} - W_t) \\ &\quad + \sum_{i=n+1}^{[T]} \frac{1}{2} f^{i-n}(\sigma_n^2, W_{n+1} - W_t + g(Z_t, Z_n, \sigma_n, t-n), W_{n+2} - W_{n+1}, \ldots, W_i - W_{i-1}) \\ &\quad - \frac{1}{2} f^{[T]-n}(\sigma_n^2, W_{n+1} - W_t + g(Z_t, Z_n, \sigma_n, t-n), W_{n+2} - W_{n+1}, \ldots, W_{[T]} - W_{[T]-1}) \\ &\quad ([T] + 1 - T) \\ &\quad + \sum_{i=n+1}^{[T]} \Big(f^{i-n}(\sigma_n^2, W_{n+1} - W_t + g(Z_t, Z_n, \sigma_n, t-n), W_{n+2} - W_{n+1}, \ldots, W_i - W_{i-1}) \Big)^{\frac{1}{2}} \\ &\quad (W_{i+1} - W_i) \\ &\quad - \Big(f^{[T]-n}(\sigma_n^2, W_{n+1} - W_t + g(Z_t, Z_n, \sigma_n, t-n), W_{n+2} - W_{n+1}, \ldots, W_{[T]} - W_{[T]-1}) \Big)^{\frac{1}{2}} \\ &\quad (W_{[T]+1} - W_T) \Big] \\ &=: h(Z_t, Z_n, \sigma_n, T, W_{n+1} - W_t, W_{n+2} - W_{n+1}, \ldots, W_{[T]} - W_{[T]-1}, W_T - W_{[T]}), \end{aligned}$$

where $h : \mathbf{R}^{[T]-n+5} \to \mathbf{R}$ is measurable. Hence we can express the discounted price at time t for the claim $\psi(Z_T)$ as

$$\begin{aligned} e^{-rt}\pi_t &= E_{P^*}(e^{-rT}\psi(Z_T) | \mathcal{F}_t) \\ &= \int e^{-rT} \psi(h(Z_t, Z_n, \sigma_n, T, x_1, \ldots, x_k))\, N(\mathbf{0}, \mathbf{\Sigma}) d(x_1, \ldots, x_k), \end{aligned} \tag{2.0.4}$$

where $k = [T] - n + 1$ (this is roughly the number of remaining time periods) and $\mathbf{\Sigma} = \mathrm{diag}(n+1-t, 1, \ldots, 1, T - [T]) \in \mathbf{R}^{k \times k}$. □

Proof of Lemma 8. We start by outlining the proof. In (2.0.4), p_n is represented as $\int A\, dB$, where both A and B depend on (z,t). To prove differentiability, we want to express p_n as $\int f\, d\nu$, where ν does not depend on (z,t) and where f and its derivatives are functions of (z,t) that are majorized by a function D independent of (z,t) such that $\int D\, d\nu < \infty$.

Observe that, with the notation of the preceding proof,

$$h(Z_t, Z_n, \sigma_n, T, x_1, x_2, \ldots, x_k) = h(1, Z_n, \sigma_n, T, x_1 + \frac{1}{\sigma_n} \log Z_t, x_2, \ldots, x_k),$$

because

$$g(Z_t, Z_n, \sigma_n, t-n) = g(1, Z_n, \sigma_n, t-n) + \frac{1}{\sigma_n} \log Z_t.$$

Since the claim is assumed integrable, it follows

$$\begin{aligned}
\infty &> E_{P^*}(e^{-rT}\psi(Z_T)) \\
&= \int e^{-rT}\psi(h(Z_t, Z_n, \sigma_n, T, W_{n+1} - W_t, W_{n+2} - W_{n+1}, \ldots, W_{[T]} - W_{[T]-1}, \\
&\quad W_T - W_{[T]}))\, dP^* \\
&= \int e^{-rT}\psi(h(1, Z_n, \sigma_n, T, W_{n+1} - W_t + \frac{1}{\sigma_n}\log Z_t, W_{n+2} - W_{n+1}, \ldots, W_{[T]} - W_{[T]-1}, \\
&\quad W_T - W_{[T]}))\, dP^* \\
&= \int e^{-rT}\psi(h(1, \hat{z}, \hat{\sigma}, T, x_1, \ldots, x_k)) \\
&\quad \Big(P^*_{(W_{n+1}-W_t+\sigma_n^{-1}\log Z_t, W_{n+2}-W_{n+1}, \ldots, W_{[T]}-W_{[T]-1}, W_T - W_{[T]})|(Z_t, Z_n, \sigma_n)} \otimes P^*_{Z_t|(Z_n,\sigma_n)} \\
&\quad \otimes P^*_{(Z_n,\sigma_n)}\Big)\, d(x_1, \ldots, x_k, z, \hat{z}, \hat{\sigma}) \\
&= \int\!\!\int\!\!\int e^{-rT}\psi(h(1, \hat{z}, \hat{\sigma}, T, x_1, \ldots, x_k)) \\
&\quad P^*_{(W_{n+1}-W_t+\hat{\sigma}^{-1}\log z, W_{n+2}-W_{n+1}, \ldots, W_{[T]}-W_{[T]-1}, W_T-W_{[T]})} d(x_1, \ldots, x_k) \\
&\quad P^*_{Z_t|(Z_n,\sigma_n)=(\hat{z},\hat{\sigma})}(dz)\, P^*_{(Z_n,\sigma_n)} d(\hat{z}, \hat{\sigma}),
\end{aligned}$$

where $k = [T] - n + 1$. Fix $(\hat{z}, \hat{\sigma})$. The following holds $P^*_{(Z_n,\sigma_n)}$-almost surely:

By Fubini's theorem the innermost integral is a.e. finite. More precisely,

$$\int e^{-rT}\psi(h(1, \hat{z}, \hat{\sigma}, T, x_1, x_2, \ldots))\, P^*_{(W_{n+1}-W_t+\hat{\sigma}^{-1}\log z, W_{n+2}-W_{n+1}, \ldots)} d(x_1, \ldots, x_k)$$

is finite for $P^*_{Z_t|(Z_n,\sigma_n)=(\hat{z},\hat{\sigma})}$-almost all z, i.e., λ-almost all z, since, by (2.0.3), the probability measure $P^*_{Z_t|(Z_n,\sigma_n)=(\hat{z},\hat{\sigma})}$ is equivalent to Lebesgue measure. As this is true for all $t \in (n, n+1)$ we conclude that $e^{-rT}\psi(h(1, \hat{z}, \hat{\sigma}, T, \bullet))$ is $N(\mathbf{m}_a, \mathbf{\Sigma}_b)$-integrable with $\mathbf{m}_a := (a, 0, \ldots, 0) \in \mathbf{R}^k$, $\mathbf{\Sigma}_b := \mathrm{diag}(b, 1, \ldots, 1, T - [T]) \in \mathbf{R}^{k \times k}$ for $0 < b < 1$ and for λ-almost all a.

Now define a measure ν on $\mathbf{R}$ by its density

$$\frac{\nu(dx)}{dx} = \int e^{-rT}\psi(h(1, \hat{z}, \hat{\sigma}, T, x, x_2, \ldots, x_k)\, N(\mathbf{0}, \mathrm{diag}(1, \ldots, 1, T - [T]))d(x_2, \ldots, x_k).$$

We have already shown that any normal density with variance between 0 and 1 is integrable with respect to ν. In view of (2.0.4), we have

$$p_n(z, \hat{z}, \hat{\sigma}, t, T) = \int \frac{1}{\sqrt{2\pi(n+1-t)}} \exp\left[-\frac{1}{2}\frac{(x - \hat{\sigma}^{-1}\log z)^2}{n+1-t}\right] \nu(dx).$$

One can easily show that the first and higher order derivatives of

$$f(x,z,t) := \frac{1}{\sqrt{2\pi(n+1-t)}} \exp\left[-\frac{1}{2}\frac{(x-\hat{\sigma}^{-1}\log z)^2}{n+1-t}\right]$$

with respect to (z,t) are of the form $f(x,z,t)g(x,z,t)$, where g is a polynomial in x with coefficients that are continuous in (z,t).

Fix (z,t). Note that for any $j \in \mathbf{N}, a \in \mathbf{R}, b > 0, \delta > 0, \epsilon > 0$ there exists a $C > 0$ such that for all $\tilde{a} \in \mathbf{R}$ with $|a-\tilde{a}| < \delta$, all $\tilde{b} > 0$ with $|b-\tilde{b}| < \epsilon$, and all $x \in \mathbf{R}$ we have

$$|x|^j \exp\left(-\frac{1}{2}\frac{(x-\tilde{a})^2}{\tilde{b}}\right) < C\exp\left(-\frac{1}{2}\frac{(x-a)^2}{b+2\epsilon}\right).$$

Thus for any $\epsilon > 0$ there exists a neighbourhood of (z,t) and a $C > 0$ such that for all $(\tilde{z},\tilde{t})$ in this neighbourhood

$$|f(x,\tilde{z},\tilde{t})g(x,\tilde{z},\tilde{t})| \le C|f(x,z,t-2\epsilon)|,$$

which is integrable with respect to ν if ϵ is small enough. By iterating Theorem 16.8 in Billingsley (1986) (interchanging differentiation and integration) we conclude that p_n is C^∞ with respect to (z,t), in particular C^2. □

Proof of Theorem 9. We know from Theorem 3 that there exists an admissable strategy ψ that generates C. Hence ψ satisfies (1.2.9) and (1.2.10). We want to show that our $\phi = (\phi^0, \phi^1)$ satisfies these conditions as well and $V(\phi) = V(\psi)$. Note first that $V_t(\phi) = e^{-rt}\pi_t = V_t(\psi)$ by the definition of ϕ and because ψ generates C.

Although, by Lemma 8, p_n may not be a C^2-function in all its arguments (namely Z_n and σ_n), we can still apply the Itô-formula, because, for $t \in (n, \min(n+1,T))$, the random variables Z_n and σ_n do not depend on t (as long as n is fixed). For t in this interval, we obtain by (1.2.10), $\psi_t^1\, dZ_t = dV_t(\psi) = d(e^{-rt}\pi_t)$, that is,

$$\begin{aligned}\psi_t^1\, dZ_t &= D_1 p_n(Z_t, Z_n, \sigma_n, t, T)\, dZ_t + D_4 p_n(Z_t, Z_n, \sigma_n, t, T)\, dt \\ &\quad + \frac{1}{2} D_{11} p_n(Z_t, Z_n, \sigma_n, t, T)\, d[Z,Z]_t,\end{aligned}$$

where D_i and D_{ii} denote partial derivatives. Hence the process

$$U_\bullet = \int_{n+}^{\bullet} (\psi_t^1 - D_1 p_n(Z_t, Z_n, \sigma_n, t, T))\, dZ_t$$

is a finite variation process. Since Z is a continuous local martingale (with respect to P^*), the process $U_\bullet$ is, moreover, a continuous local martingale. It must be identically 0, because a continuous local martingale with bounded variation is constant (Protter (1990), Theorem III.1.3). It follows that $\int_{n+}^{\bullet} \psi_t^1\, dZ_t = \int_{n+}^{\bullet} D_1 p_n(Z_t, Z_n, \sigma_n, t, T)\, dZ_t$. Considering the quadratic variation of both sides, we also have

$$\int_{n+}^{\bullet} (\psi_t^1)^2\, d[Z]_t = \int_{n+}^{\bullet} (D_1 p_n(Z_t, Z_n, \sigma_n, t, T))^2\, d[Z]_t.$$

Conditions (1.2.9) and (1.2.10) for ϕ now follow from the same conditions for ψ. □

Chapter 3

Equality of filtrations

In this chapter, we give a proof of Lemma 2 which states that the filtrations generated by the processes B, W and S of Section 1.2 are all equal. While one could do this under the specific assumptions of that section, it is more instructive to consider arbitrary locally functional Lipschitz operators. For a more general but somewhat different context, see Kallsen & Taqqu (1994).

3.1 Locally functional Lipschitz operators

We fix throughout a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions, i.e., $\mathcal{F}_0$ contains all null sets of P and $(\mathcal{F}_t)_{t\geq 0}$ is right continuous.

The following notation will be used: $\|\cdot\|$ (without sub- and superscripts) is the Euclidean norm (in $\mathbf{R}^d$). D^d denotes the set of (non-random) functions $f : \mathbf{R}^+ \to \mathbf{R}^d$ that are càdlàg. $\mathbf{D}^d$ stands for the set of adapted càdlàg processes in $\mathbf{R}^d$. We let $\mathcal{S}^d \in \mathbf{D}^d$ denote the space of semimartingales in $\mathbf{R}^d$. For a process $X \in \mathbf{D}^d$, small superscripts refer to components and greek superscripts to stopping times. Thus X^τ means the process X stopped at time τ; X^k and $(X^k)^\tau$ mean respectively the k-th coordinate process of X and that coordinate process stopped at τ. Stochastic integrals $Z = \int Y\, dX$, where X is $\mathbf{R}^{d_1}$- and Y is $\mathbf{R}^{d_2\times d_1}$-valued, are to be interpreted in the sense of matrix multiplication, i.e., $Z^j = \sum_{k=1}^{d_1} \int Y^{jk}\, dX^k$, $j = 1, \ldots, d_2$.

Let τ be a predictable stopping time. We say that a function $X : [0, \tau[\to \mathbf{R}^d$ is a *semimartingale on* $[0,\tau[$ if $X^{\overline{\tau}}$ (i.e., the mapping $(X_{\overline{\tau}\wedge t})_{t\geq 0} : \mathbf{R}^+ \times \Omega \to \mathbf{R}^d$) is a semimartingale for any stopping time $\overline{\tau} < \tau$. Let $(\mathcal{G}_t)_{t\geq 0}$ be a sub-filtration of $(\mathcal{F}_t)_{t\geq 0}$ (i.e., $\mathcal{G}_t \subset \mathcal{F}_t$, $t \geq 0$). We call a semimartingale X on $[0,\tau[$ *adapted* to $(\mathcal{G}_t)_{t\geq 0}$ if $X^{\overline{\tau}}$ is $(\mathcal{G}_t)_{t\geq 0}$-adapted for any stopping time $\overline{\tau} < \tau$. We call an operator $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ *adapted* to $(\mathcal{G}_t)_{t\geq 0}$ if the process $F(X)$ is $(\mathcal{G}_t)_{t\geq 0}$-adapted for any $(\mathcal{G}_t)_{t\geq 0}$-adapted process $X \in \mathbf{D}^{d_1}$.

Recall that a function $f : \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ is called *locally Lipschitz* if for any $x_0 \in \mathbf{R}^{d_1}$ there is an open neighbourhood U of x_0 and a constant $K > 0$ such that for all $x, y \in U$ we have

$$\|f(x) - f(y)\| \leq K\|x - y\|.$$

The corresponding definition for operators acting on stochastic processes takes a slightly different form: We call an operator $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ *locally functional Lipschitz* if the following holds:

(i) for any $X, Y \in \mathbf{D}^{d_1}$ and any stopping time τ, $X^{\tau-} = Y^{\tau-}$ implies $F(X)^{\tau-} = F(Y)^{\tau-}$,

(ii) for any $b > 0$ there exists an increasing (finite) process $(K_t)_{t\geq 0}$ such that for any $X, Y \in \mathbf{D}^{d_1}$ with $\sup_{t\geq 0} \|X_t\| < b$ and $\sup_{t\geq 0} \|Y_t\| < b$, and for any $t \geq 0$ we have

$$\|F(X)_t - F(Y)_t\| \leq K_t \sup_{0\leq s\leq t} \|X_s - Y_s\| \text{ a.s.}$$

If the condition $\sup_{t\geq 0} \|X_t\| < b$ and $\sup_{t\geq 0} \|Y_t\| < b$, in (ii) above, is suppressed, then we say that $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ is *functional Lipschitz*.

3.2 Stability properties

The first proposition shows how to commonly obtain locally functional Lipschitz operators.

Proposition 10 *1. If $f : \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ has continuous first partial derivatives, or more generally, is locally Lipschitz, then the operator $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$, $F(X)(\omega, t) := f(X(\omega, t))$, is locally functional Lipschitz and adapted to any sub-filtration of $(\mathcal{F}_t)_{t\geq 0}$.*

2. Any process $U \in \mathbf{D}^{d_2}$ (viewed as an operator $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ of the form $F(\cdot) = U$), and in particular any constant, is functional Lipschitz and hence locally functional Lipschitz.

Proof.

1. Functions with continuous first partial derivatives are locally Lipschitz (e.g. Lang (1968), p. 409). Assume therefore that f is locally Lipschitz. The operator $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ is well-defined because the image of an adapted càdlàg process under a continuous function is again an adapted càdlàg process (e.g. Protter (1990), Theorem III.7.33). Condition (i) in the definition of "locally functional Lipschitz" is satisfied because F involves here a pointwise transformation. Using a compactness argument we conclude that $f|_B$ is *Lipschitz* for any compact set B, i.e., there exists a *Lipschitz constant* $K > 0$ such that for any $x, y \in B$ we have $\|f(x) - f(y)\| \leq K\|x - y\|$. Fix $b > 0$. Let K be the Lipschitz constant for the choice $B = \overline{B}_n := \{x \in \mathbf{R}^{d_1} : \|x\| \leq b\}$. Condition (ii) then holds with $K_t := K$ for all $t \geq 0$.

2. (i) and (ii) in the definition of "(locally) functional Lipschitz" hold trivially, since $F(X)$ does not depend on X. □

The following proposition shows that the locally functional Lipschitz condition is closed under several commonly used operations. Its statements can be read with or without the inclusion of the parentheses involving adaptedness.

Proposition 11 *1. $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ is locally functional Lipschitz (and adapted to a sub-filtration $(\mathcal{G}_t)_{t\geq 0}$) if and only if all $F^j : \mathbf{D}^{d_1} \to \mathbf{D}^1$, $j = 1, \ldots, d_2$, are locally functional Lipschitz (and adapted to $(\mathcal{G}_t)_{t\geq 0}$).*

2. *If $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ and $G : \mathbf{D}^{d_2} \to \mathbf{D}^{d_3}$ are locally functional Lipschitz (and F, G are adapted to a sub-filtration $(\mathcal{G}_t)_{t\geq 0}$) then the composition $G \circ F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_3}$ is locally functional Lipschitz (and adapted to $(\mathcal{G}_t)_{t\geq 0}$) as well.*

3. *If $F, G : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ are locally functional Lipschitz (and adapted to a sub-filtration $(\mathcal{G}_t)_{t\geq 0}$), so is $F+G$ and, for $d_2 = 1$, so is FG (in the sense of pointwise multiplication).*

   If $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2\times d_2}$ is locally functional Lipschitz (and adapted to a sub-filtration $(\mathcal{G}_t)_{t\geq 0}$), and if there is a decreasing function $L : \mathbf{R}^+ \to \mathbf{R}^+\backslash\{0\}$ with $\det(F(X)(\omega, t)) > L(t)$ for all X, ω, t, then the pointwise matrix inverse $F^{-1} : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2\times d_2}$ is locally functional Lipschitz (and adapted to $(\mathcal{G}_t)_{t\geq 0}$). In particular, if $F : \mathbf{D}^{d_1} \to \mathbf{D}^1$ is locally functional Lipschitz and $F > c \in (0,\infty)$, then $1/F : \mathbf{D}^{d_1} \to \mathbf{D}^1$ is locally functional Lipschitz.

Proof. The statements have two versions, the second involving the adaptedness of an operator F. We leave the easy adaptedness part of the proof to the reader.

1. This is evident.

2. Condition (i) holds for $G \circ F$, since $X^{\tau-} = Y^{\tau-}$ implies $F(X)^{\tau-} = F(Y)^{\tau-}$ which itself implies $G(F(X))^{\tau-} = G(F(Y))^{\tau-}$. Let $b > 0$. In order to show (ii) it suffices to show that for any $r > 0$ there exists an increasing (finite) process $(L_t)_{0\leq t\leq r}$ such that for any $X, Y \in \mathbf{D}^{d_1}$ with $\sup_{t\geq 0} \|X_t\| < b$ and $\sup_{t\geq 0} \|Y_t\| < b$, and for any $t \leq r$ we have

   $$\|F(X)_t - F(Y)_t\| \leq L_t \sup_{0\leq s\leq t} \|X_s - Y_s\| \text{ a.s.}$$

   Let $r > 0$. Let $(K_t^{F,b})_{t\geq 0}$ be a process to F and b as in (ii) of the definition of "locally functional Lipschitz." It follows that for any $X \in \mathbf{D}^{d_1}$ with $\sup_{t\geq 0} \|X_t\| < b$ we have

   $$\begin{aligned}
   \sup_{0\leq t\leq r+1} \|F(X)_t\| &\leq \sup_{0\leq t\leq r+1} \|F(0)_t\| + \sup_{0\leq t\leq r+1} \|F(X)_s - F(0)_s\| \\
   &\leq \sup_{0\leq t\leq r+1} \|F(0)_t\| + K_{r+1} \sup_{0\leq t\leq r+1} \|X_s\| \\
   &\leq \sup_{0\leq t\leq r+1} \|F(0)_t\| + K_{r+1} b \qquad =: c
   \end{aligned}$$

   Let $(K_t^{G,c})_{t\geq 0}$ be a process to G and c as in (ii) of the definition of "locally functional Lipschitz" and define the increasing process $(L_t)_{0\leq t\leq r}$ by $L_t := K_t^{F,b} K_t^{G,c}$ for all $t \leq r$. Fix $t \leq r$ and $X, Y \in \mathbf{D}^{d_1}$ with $\sup_{t\geq 0} \|X_t\| < b$ and $\sup_{t\geq 0} \|Y_t\| < b$. Defining the processes $\widetilde{X}, \widetilde{Y} \in \mathbf{D}^{d_2}$ by $\widetilde{X}_t := F(X)_{t\wedge(r+1)}$, $\widetilde{Y}_t := F(Y)_{t\wedge(r+1)}$ for all $t \geq 0$, we have $\sup_{t\geq 0} \|\widetilde{X}_t\| < c$, $\sup_{t\geq 0} \|\widetilde{Y}_t\| < c$. It follows for any $t \geq r$:

   $$\begin{aligned}
   \|G \circ F(X)_t - G \circ F(Y)_t\| &= \|G(\widetilde{X})_t - G(\widetilde{Y})_t\| \\
   &\leq K_t^{G,c} \sup_{0\leq s\leq t} \|\widetilde{X}_s - \widetilde{Y}_s\|
   \end{aligned}$$

$$\begin{aligned} &\leq K_t^{G,c} \sup_{0\leq s\leq t} \|F(X)_s - F(Y)_s\| \\ &\leq K_t^{G,c} K_t^{F,b} \sup_{0\leq s\leq t} \|X_s - Y_s\| \\ &\leq L_t \sup_{0\leq s\leq t} \|X_s - Y_s\|. \end{aligned}$$

3. By Statement 1 the mapping $(F, G) : \mathbf{D}^{d_1} \to \mathbf{D}^{d_1+d_2}$ is locally functional Lipschitz if F, G are locally functional Lipschitz. The functions $\oplus : \mathbf{R}^{d_2+d_2} \to \mathbf{R}^{d_2}, (x_1, x_2) \mapsto x_1 + x_2$ and $\otimes : \mathbf{R}^2 \to \mathbf{R}, (x_1, x_2) \mapsto x_1x_2$ have continuous partial derivatives. It follows from Proposition 10.1 that the induced operators $\oplus : \mathbf{D}^{d_2+d_2} \to \mathbf{D}^{d_2}$ and $\otimes : \mathbf{D}^2 \to \mathbf{D}^1$ are locally functional Lipschitz. Applying Statement 2 we conclude that $F + G = (\oplus) \circ (F, G) : \mathbf{D}^{d_1} \to \mathbf{D}^{d_2}$ and $FG = (\otimes) \circ (F, G) : \mathbf{D}^1 \to \mathbf{D}^1$ are locally functional Lipschitz.

   In order to prove the statement concerning F^{-1} we may assume that $d_2 = 1$: Otherwise note that by Cramer's rule the inverse of a matrix $A \in \mathbf{R}^{d_2\times d_2}$ has components of the form $s/\det A$ where both s and $\det A$ are sums of products of components of A. In view of 1., 3. (for sums and products), and 2. it suffices to prove the statement for $d_2 = 1$.

   Condition (i) in the definition of "locally functional Lipschitz" is satisfied for F^{-1}. For the proof of Condition (ii) observe that for any $b > 0$, for any $X, Y \in \mathbf{D}^{d_1}$ with $\sup_{t\geq 0} \|X_t\| < b$ and $\sup_{t\geq 0} \|Y_t\| < b$, and for any $t \geq 0$ we have

$$\begin{aligned} \left|\frac{1}{F(X)_t} - \frac{1}{F(Y)_t}\right| &= \left|(F(Y)_t - F(X)_t)\Big(\frac{1}{F(X)_t F(Y)_t}\Big)\right| \\ &\leq L(t)^{-2} \|F(Y)_t - F(X)_t\| \\ &\leq L(t)^{-2} K_t \sup_{0\leq s\leq t} \|X_s - Y_s\| \end{aligned}$$

   where $(K_t)_{t\geq 0}$ is defined as in the definition of "locally functional Lipschitz." □

3.3 Existence, pathwise uniqueness and adaptedness

We now turn to the problem whether the solution to a stochastic differential equation (SDE) is adapted to a given filtration. Under a locally functional Lipschitz condition we get this adaptedness as a by-product of an existence and uniqueness theorem that can be found in Métivier (1982) or in slightly different versions in Métivier & Pellaumail (1980) and Jacod (1979).

Consider the following stochastic differential equation:

$$X_t = J_t + \int_0^t F(X)_{s-} \, dZ_s. \tag{3.3.1}$$

We are interested in sufficient conditions for J, Z, and F that guarantee the existence, pathwise uniqueness and adaptedness of a solution X.

Theorem 12 *Let J be a semimartingale in $\mathbf{R}^{d_1}$, Z a semimartingale in $\mathbf{R}^{d_2}$ with $Z_0 = 0$, and $F : \mathbf{D}^{d_1} \to \mathbf{D}^{d_1 \times d_2}$ a locally functional Lipschitz operator.*

1. *Then there exists a predictable stopping time τ and a semimartingale X on $[0, \tau[$ with values in $\mathbf{R}^{d_1}$ such that*

   (a) $\limsup_{t \uparrow \tau} \|X_t\| = \infty$ *on* $\{\tau < \infty\}$ *a.s.*

   (b) X solves the SDE (3.3.1) on $[0, \tau[$, *i.e., for any stopping time $\overline{\tau} < \tau$ we have*

$$X_t^{\overline{\tau}} = J_t^{\overline{\tau}} + \int_0^{\overline{\tau} \wedge t} F(X)_{s-} \, dZ_s. \tag{3.3.2}$$

2. *We have* pathwise uniqueness *of the solution X, i.e., if τ' is a predictable stopping time and X' is a semimartingale on $[0, \tau'[$ such that (a) and (b) hold with τ', X' instead of τ, X then $\tau = \tau'$ a.s. and X, X' are indistinguishable.*

3. *If F is functional Lipschitz, then $\tau = \infty$ a.s.*

4. *Let, moreover, $(\mathcal{G}_t)_{t \geq 0}$ be any P-complete sub-filtration of $(\mathcal{F}_t)_{t \geq 0}$. If J, Z and F are $(\mathcal{G}_t)_{t \geq 0}$-adapted ($F$ in the sense of Section 3.1) then the unique solution X from Statement 1 is adapted to $(\mathcal{G}_t)_{t \geq 0}$ as well.*

Proof. Statements 1,2 and 3 follow from Theorem 34.7, Corollary 35.3, and Remark 35.4 in Métivier (1982).

Let J, Z and F be adapted to a P-complete sub-filtration $(\mathcal{G}_t)_{t \geq 0}$. If we replace the original filtration by $(\mathcal{G}_t)_{t \geq 0}$ (and restrict F to the set of $(\mathcal{G}_t)_{t \geq 0}$-adapted processes) it follows that Equation (3.3.1) has a solution X' on $[0, \tau'[$, where τ' is a $(\mathcal{G}_t)_{t \geq 0}$-predictable stopping time (and hence also a $(\mathcal{F}_t)_{t \geq 0}$-predictable stopping time) and X' a $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t \geq 0}, P)$-semimartingale on $[0, \tau'[$. Note that X' is also adapted to the bigger filtration $(\mathcal{F}_t)_{t \geq 0}$. Since $X' \in \mathbf{D}^{d_1}$, the right-hand side of Equation (3.3.2) with (X', τ') instead of (X, τ) is a semimartingale w.r.t. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. It follows that X' is also a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$-semimartingale. From 2. we conclude that $\tau' = \tau$ a.s. and X, X' are indistinguishable. Hence X is adapted to $(\mathcal{G}_t)_{t \geq 0}$. □

3.4 Equality of the filtrations in the ARCH models

We can now return to the setting of Section 1.2 and prove Lemma 2.

Theorem 13 *The P-completed filtrations generated by either B, W or S coincide.*

Proof. The processes are defined on $[0, T]$ where T is a fixed real number whereas the results of Chapter 3 apply to processes defined for all $t \geq 0$. To apply these results we replace any process $(X_t)_{0 \leq t \leq T}$ by $(X_{t \wedge T})_{t \geq 0}$ and accordingly for filtrations and operators.

Since by construction W and S are adapted to the given filtration, which is the natural filtration of B, it suffices to show that B is adapted to the P-completed filtration generated by W and that generated by S.

First we prove that B is adapted to the P-completed natural filtration of W. To show this, we express X as a solution to a SDE with W as integrator. From Equation (1.2.8) we have

$$B_t = \int_0^t -\frac{\mu(F(X)_{s-}) - r}{F(X)_{s-}}\, ds + \int_0^t 1\, dW_s, \tag{3.4.1}$$

and from Equation (1.2.2) it follows

$$X_t = \int_0^t -(\mu(F(X)_{s-}) - r)\, ds + \int_0^t F(X)_{s-}\, dW_s.$$

The functional Lipschitz operator F defined in Section 1.2 induces a functional Lipschitz and hence locally functional Lipschitz operator $\widetilde{F} : \mathbf{D} \to \mathbf{D}$ that is adapted to any P-complete sub-filtration of $(\mathcal{F}_t)_{t\geq 0}$. Let $\widetilde{\mu} : \mathbf{R} \to \mathbf{R}$ be a differentiable continuation of $\mu : \mathbf{R}^+ \to \mathbf{R}$ with continuous derivative. The induced operator $\widetilde{\mu} : \mathbf{D} \to \mathbf{D}$ is locally functional Lipschitz by Proposition 10.1. Applying Proposition 11 and Statement 2 of Proposition 10 shows that the operator

$$J : \mathbf{D} \to \mathbf{D}^2, \quad Z \mapsto (-(\mu(F(Z)_s) - r), F(Z)_s),$$

is locally functional Lipschitz as well, and we have

$$X_t = \int_0^t J^1(X)_{s-}\, ds + \int_0^t J^2(X)_{s-}\, dW_s.$$

Since $\widetilde{F}$ is adapted to all P-complete sub-filtrations of $(\mathcal{F}_t)_{0\leq t\leq T}$ we conclude that J has this property as well. We can now apply Theorem 12 with $\overline{X}_t := X_t$, $\overline{J}_t := 0$, $\overline{Z}_t := (t, W_t)$, $\overline{F} := J$, and conclude that X is adapted to the P-completed filtration generated by W. Now, since all the processes on the right hand side of Equation (3.4.1) are adapted to the P-completed natural filtration of W, the same is true for B.

We finally prove that B is adapted to the P-completed natural filtration of S. As in the first part, we express X as a solution to a SDE with S as integrator. From Equation (1.2.4) we know that S^1 solves the SDE

$$S_t^1 = S_0^1 + \int_0^t \mu(\sigma_{s-}) S_{s-}^1\, ds + \int_0^t \sigma_{s-} S_{s-}^1\, dB_s.$$

Hence,

$$\begin{aligned} B_t &= \int_0^t \frac{1}{\sigma_{s-} S_{s-}^1} \sigma_{s-} S_{s-}^1\, dB_s \\ &= -\int_0^t \frac{\mu(F(X)_{s-})}{F(X)_{s-}}\, ds + \int_0^t \frac{1}{F(X)_{s-} S_{s-}^1}\, d(S^1 - S_0^1)_s, \end{aligned} \tag{3.4.2}$$

and from Equation (1.2.2) we get

$$\begin{aligned} X_t &= \int_0^t -\mu(F(X)_{s-})\, ds + \int_0^t \frac{1}{S_{s-}^1}\, d(S^1 - S_0^1)_s \\ &= \int_0^t K^1(X)_{s-}\, ds + \int_0^t K^2(X)_{s-}\, d(S^1 - S_0^1)_s, \end{aligned}$$

where we denote

$$K : \mathbf{D} \to \mathbf{D}^2, \quad Z \mapsto (-\mu(F(Z)_s), 1/S_s^1)$$

As in the first part of the proof, K is a locally functional Lipschitz operator adapted to the P-completed filtration generated by S. Applying Theorem 12, with $\overline{X}_t := X_t$, $\overline{J}_t := 0$, $\overline{Z}_t := (t, S_t^1 - S_0^1)$, $\overline{F} := K$, we conclude that X is adapted to the P-completed filtration generated by S. As above we conclude from Equation (3.4.2) that B is adapted to the P-completed natural filtration of S as well. $\square$

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