Modelling Financial Data Using Generalized Hyperbolic Distributions

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Abstract

This note describes estimation algorithms for generalized hyperbolic, hyperbolic and normal inverse Gaussian distributions. These distributions provide a better fit to empirically observed log-return distributions of financial assets than the classical normal distributions. Based on the better fit to the semi-heavy tails of financial assets we can compute more realistic Value-at-Risk estimates.

The modelling of financial assets as stochastic processes is determined by distributional assumptions on the increments and the dependence structure. It is well known that the returns of most financial assets have semi-heavy tails, i.e. the actual kurtosis is higher than the zero kurtosis of the normal distribution (see Pagan (1996)). On the other hand the use of stable distributions leads to models with nonexisting moments.

The class of generalized hyperbolic distributions and its sub-classes – the hyperbolic and the normal inverse Gaussian distributions – possess these semi-heavy tails. Generalized hyperbolic distributions were introduced by Barndorff-Nielsen (1977) and applied e.g. to model grain size distributions of wind blown sands. The mathematical properties of these distributions are well-known (see Barndorff-Nielsen/Blæsild (1981)). Recently generalized hyperbolic distributions resp. their sub-classes were proposed as a model for the distribution of increments of financial price processes (see Eberlein/Keller (1995), Rydberg (1996), Barndorff-Nielsen (1998), Eberlein/Keller/Prause (1997)) and as limit distributions of diffusions (see Bibby/Sørensen (1997)). Nevertheless studies were only published concerning the estimation and application to financial data in the special case of hyperbolic distributions. In this study we present parameter estimations for German stock and US stock index data and evaluate the goodness of fit. In particular we look at the tails of the distributions.

1 Generalized Hyperbolic Distributions

Generalized hyperbolic (GH) distributions are given by the Lebesgue density

$$gh(x;\lambda,\alpha,\beta,\delta,\mu) = a_\lambda\big(\delta^2+(x-\mu)^2\big)^{(\lambda-1/2)/2}\,\mathrm{K}_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\,\big)\exp\big(\beta(x-\mu)\big) \tag{1}$$

$$a_\lambda = a_\lambda(\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda\,\mathrm{K}_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\,\big)}; \qquad x,\mu\in\mathbf{R},$$

$$\begin{aligned} &\delta \ge 0,\ |\beta|<\alpha \quad \text{if } \lambda>0\\ &\delta > 0,\ |\beta|<\alpha \quad \text{if } \lambda=0\\ &\delta > 0,\ |\beta|\le\alpha \quad \text{if } \lambda<0, \end{aligned} \tag{2}$$

where K_λ is a modified Bessel function. The parameters μ and δ describe the location and the scale of the distribution. Note that this distribution may be represented as a normal variance-mean mixture with the generalized inverse Gaussian as mixing distribution (see Barndorff-Nielsen/Blæsild (1981)). The normal distribution is obtained as a limiting case for $\delta\to\infty$ and $\delta/\alpha\to\sigma^2$ (see Barndorff-Nielsen (1978)). Generalized hyperbolic distributions are infinitely divisible, hence they generate a Lévy processes (see Barndorff-Nielsen/Halgreen (1977), Eberlein/Keller (1995)).

Using the properties of Bessel functions K_λ it is possible to simplify the function gh whenever $\lambda=-0.5$, 0, 0.5 or 1. For $\lambda=-0.5$ we get the normal inverse Gaussian (NIG) distribution

$$nig(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\exp\big(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\big)\frac{\mathrm{K}_1\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\,\big)}{\sqrt{\delta^2+(x-\mu)^2}} \tag{3}$$

$$x,\mu\in\mathbf{R},\ 0\le\delta,\ |\beta|<\alpha$$

and for $\lambda=1$ the hyperbolic distribution (HYP)

$$hyp(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\delta\alpha\,\mathrm{K}_1\big(\delta\sqrt{\alpha^2-\beta^2}\,\big)}\exp\big(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\big) \tag{4}$$

$$x,\mu\in\mathbf{R},\ 0\le\delta,\ |\beta|<\alpha.$$

One drawback of using hyperbolic distributions instead of the normal distribution is that the meaning of the parameters seems to be obscure. Different parametrizations of the generalized hyperbolic distribution have been proposed to circumvent this problem

$$\xi = \big(1+\delta\sqrt{\alpha^2-\beta^2}\,\big)^{-1/2}, \qquad \chi = \xi\beta/\alpha \tag{5}$$

$$\zeta = \delta\sqrt{\alpha^2-\beta^2}, \qquad \rho = \beta/\alpha. \tag{6}$$

In the case of hyperbolic distributions the parameters (χ,ξ) may be plotted in a shape triangle, which reflects asymptotically the shape, i.e. skewness and kurtosis of the distribution (see Barndorff-Nielsen et al. (1985)).

We restrict this study to the sub-classes given above because the hyperbolic law is the fastest to estimate (see Section 2) and the NIG law is closed under convolution.

2 Estimation Algorithm

In order to estimate GH distributions we assume independent observations and maximize the log-likelihood function. We choose a numerical estimation procedure mainly based on an optimization for each coordinate. For the optimization step in one direction we implemented a refined bracketing method (see Thisted (1988), Jarrat (1970)) which makes no use of derivatives. This gives us the possibility to replace the likelihood function easily by different metrics (see Section 6), but the resulting algorithm is not as fast as a method based on derivatives could be. It was necessary to adapt the algorithm to the parameter restrictions given above. In contrast to the hyperbolic case the estimation of GH parameters for financial return data converges quite often to limit distributions at the boundary of the parameter space. Moreover, we modified the algorithm to estimate parameters for a given constant sub-class characterized by λ.

Although the computational power increases it is necessary to find a reasonable tradeoff between the introduction of additional parameters and the possible improvement of the fit. Barndorff-Nielsen/Blæsild (1981) mentioned the flatness of the likelihood function yet for the hyperbolic distribution. The change in the likelihood function of the GH distribution is even smaller for a wide range of parameters (see Section 5 below). Consequently the generalized hyperbolic distribution as a model for financial data leads to overfitting. This will become clearer in the following sections. The first four moments of return distributions yield simple and useful econometric interpretations: trend, riskiness, asymmetry and the probability of extreme events. Therefore it seems to be appropriate to model return data with one of the sub-classes which has four parameters.

Because of the restrictions on the parameter values and the flatness of the likelihood function it is not possible to use standard minimization algorithms. These ready implemented routines (see Press et al. (1992)) often assume that the parameters and the value of the function have the same order and that the gradient is not too small. Although we have no theoretically guaranteed convergence of our algorithm, the tests with different start values reveal that for financial data the use of reasonable start values results in convergence to a global extremum. In the case of hyperbolic distributions we estimate the same parameters with our algorithm and the `hyp` program implemented by Blæsild/Sørensen (1992).

The Bessel functions are calculated by a numerical approximation (see Press et al. (1992)). Note that for $\lambda = 1$ this function appears only in the norming constant. For a data set with n independent observations we need to evaluate $n+1$ Bessel functions for NIG and GH distributions and only one for $\lambda = 1$. This leads to a striking reduction in the time necessary to calculate the likelihood function in the hyperbolic case.

3 Results of the Estimation

We applied the estimator to log-return data from the German stock market and to New York Stock Exchange (NYSE) indices. The stock data set consists of daily closing prices from January 1988 to May 1994. We had to correct these quoted prices due to dividend payments. The NYSE indices are reported from January 2, 1990 to November 29, 1996. In the Tables 5 and 6 we present the estimated GH, NIG and hyperbolic distributions. The tables contain also the log-likelihood function and the second and third parametrizations (χ, ξ) and (ρ, ζ).

The estimation for λ ranges from -2.4 to 0.8 but for 23 of 30 stocks in the DAX we get $-2 < \lambda < -1.4$. In these cases the following sub-class of the generalized hyperbolic distribution with $\lambda = -3/2$ could be justified empirically

$$\widetilde{h}(x;\alpha,\beta,\delta,\mu) = \frac{a_{-3/2}}{\delta^2 + (x-\mu)^2}\,\mathrm{K}_2\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\,\big)\exp\big(\beta(x-\mu)\big) \tag{7}$$

$$a_{-3/2} = \left(\frac{\delta^2}{\alpha^2-\delta^2}\right)^{3/4}\frac{\alpha^2}{\mathrm{K}_{3/2}\big(\delta\sqrt{\alpha^2-\beta^2}\,\big)}.$$

The disadvantage of this sub-class is that it is not closed under convolution and that the estimation is time consuming because of the Bessel function outside the norming constant. Therefore we have not applied this distribution in this study.

The variation in the likelihood function for the GH distribution and the sub-classes is very small. However the comparison of the sub-classes yields a clear result: for all data sets the normal inverse Gaussian density has a higher likelihood than the hyperbolic distribution.

For seven German stocks (Allianz-Holding, Bayerische Vereinsbank, Commerzbank, Karstadt, MAN, Mannesmann, Siemens) and the NYSE Composite Index the GH distribution converges to the boundary of the parameter space as $\beta \to \alpha$, $\lambda < 0$, $0 < \delta$. In terms of the other parametrizations this means χ, ξ, $\rho \to 1$ and $\zeta \to 0$. The limit distribution has the following form

$$\bar{h}(x;\lambda,\alpha,\delta,\mu) = \frac{2^{\lambda+1}}{\sqrt{2\pi}\,\Gamma(-\lambda)\;\delta^{2\lambda}\;\alpha^{\lambda-1/2}}\,\mathrm{K}_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\,\big)\;\exp\big(\alpha(x-\mu)\big) \tag{8}$$

This limit distribution is calculated using the well-known properties of the modified Bessel function $\mathrm{K}_\nu(x) = K_{-\nu}(x)$ and $\mathrm{K}_\nu(x) \sim \Gamma(\nu)2^{\nu-1}x^{-\nu}$ for $x \downarrow 0$, $\nu > 0$ (see Abramowitz/Stegun (1968)). The parametrization in this limit case is 4-dimensional but a substantial change appears only in the norming constant.

4 Comparison of the Fits

The aim of this study is to evaluate the fit of the generalized hyperbolic distributions and their sub-classes. For a first graphical comparison we show plots of the densities and qq-plots for the NYSE Industrial Index and Bayer in Figure 1. Clearly, generalized hyperbolic distributions are leptokurtic, i.e. the peak in the centre is higher and there is more mass in the tails than for the normal distribution.

We also compare the estimates with fitted normal distributions. As a measure for the goodness of the fit we used various distances between the fitted and the empirical cumulative density function (cdf). The Kolmogorov distance is defined as the supremum over the absolute differences between two cumulative density functions. We also compute L^1 and L^2 distances of the cumulative density functions. The Anderson & Darling statistic is given by

$$AD = \max_{x\in\mathbf{R}} \frac{|F_{emp}(x) - F_{est}(x)|}{\sqrt{F_{est}(x)(1-F_{est}(x))}} \tag{9}$$

where F_{emp} and F_{est} are the empirical and the estimated cdf. We use this statistic because it pays more attention to the tails of the distribution (see Hurst, Platen, Rachev (1995)) and therefore hints at the possibility to model the probability of extreme events with a given distribution. In Table 1 we give the results for the some share values of the German DAX.

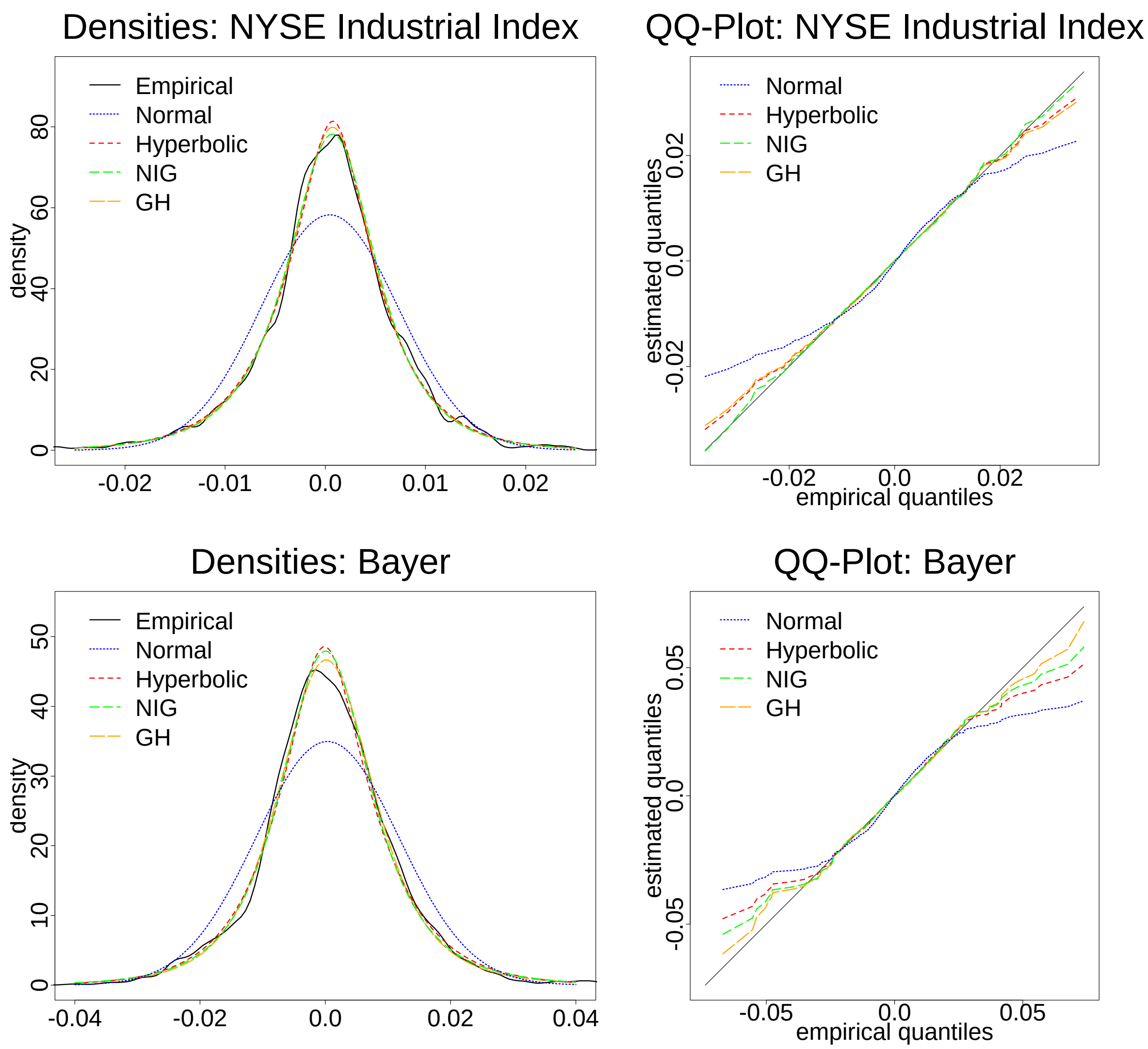

Figure 1: *Density and qq-plots of the returns of NYSE Industrial Index and Bayer.*

For all the analyzed metrics we get better results for the GH distributions and their subclasses than for the normal distribution. The poor fit of the normal distribution to the semi-heavy tails is obvious from the values of the Anderson & Darling statistic. Looking at the statistics for the GH, NIG and HYP distributions we find no striking differences. Because of the flatness of the likelihood function and the proximity of the log-likelihood values in Tables 5 and 6 this result is no surprise and underlines the overfitting of the generalized hyperbolic distribution. The values of the Kolmogorov and L^2 distances of the GH, NIG and HYP are very close and the distribution with the highest value changes. The Anderson & Darling statistic and the L^1 distances reveal the following ranks in the goodness of fit: GH, NIG, hyperbolic and normal distribution.

	Kolmogorov Distance				L^2-Distance			
	GH	NIG	HYP	Normal	GH	NIG	HYP	Normal
Allianz-Holding	0.0329	0.0290	0.0225	0.0683	0.0016	0.0018	0.0019	0.0097
BASF	0.0164	0.0150	0.0136	0.0524	0.0010	0.0012	0.0014	0.0068
Bayer	0.0164	0.0167	0.0160	0.0593	0.0011	0.0012	0.0015	0.0070
BHW	0.0220	0.0225	0.0227	0.0637	0.0015	0.0014	0.0017	0.0079
BMW	0.0228	0.0229	0.0222	0.0713	0.0010	0.0013	0.0017	0.0087
Commerzbank	0.0319	0.0295	0.0269	0.0581	0.0017	0.0016	0.0015	0.0072
Continental	0.0235	0.0246	0.0247	0.0526	0.0015	0.0015	0.0016	0.0071
Daimler Benz	0.0122	0.0122	0.0120	0.0628	0.0014	0.0013	0.0014	0.0085
	Anderson & Darling Statistic				L^1-Distance			
	GH	NIG	HYP	Normal	GH	NIG	HYP	Normal
Allianz-Holding	0.1301	0.5426	3.0254	$5.84e07$	0.0004	0.0006	0.0007	0.0024
BASF	0.0674	0.2621	0.9902	$9.10e05$	0.0003	0.0003	0.0004	0.0016
Bayer	0.0604	0.0884	0.1462	17.8506	0.0003	0.0003	0.0004	0.0015
BHW	0.1477	1.0116	12.7087	$2.82e14$	0.0004	0.0004	0.0005	0.0019
BMW	0.0639	0.3166	2.0842	$3.45e08$	0.0003	0.0004	0.0005	0.0022
Commerzbank	0.1096	0.5284	1.9754	$3.08e08$	0.0004	0.0004	0.0005	0.0017
Continental	0.0508	0.0713	0.1081	23.8127	0.0005	0.0005	0.0005	0.0019
Daimler Benz	0.1094	0.5533	4.0783	$6.58e09$	0.0004	0.0005	0.0005	0.0021

Table 1: *Comparison of the fits of the GH, NIG, hyperbolic and normal distributions. Different metrics are applied to measure the difference between the estimated and the empirical cumulative density functions.*

5 Simulation

In this section we are going to analyze the stability of the estimation by a simulation study. We generate random numbers from the GH distribution by the use of the quantile function and a uniform random number generator on $[0, 1]$. We produce data sets with different sample sizes n from the distributions estimated above. Note that the choice of the sampling distributions restricts the validity of the following results to financial return data sets. In Table 7 we provide the results of the simulation for Bayer. Similar results were also obtained for other sampling distributions.

In Table 7 we see that for large n the parameter λ is close to the sampling distribution. This reveals that the estimation of sub-classes characterized by λ is quite good although the difference between the sub-classes in terms of the likelihood is small. On the other hand it becomes clear that the parameters $(\alpha, \beta, \delta, \mu)$ are converging very slowly to the sampling distribution. Note that it is not possible to find financial time series at any given length without getting trouble with changes of regime. Due to the overfitting it is not useful to compare the parameters of the

Sample Size	Kolmogorov Distance	Anderson & Darling Statistic
50	0.046873	2748.095
100	0.028267	10.383349
125	0.042437	0.088472
150	0.013544	0.032380
175	0.005031	0.025219
200	0.021564	0.050546
350	0.021612	0.049411
500	0.020363	0.051141
1000	0.018077	0.050193
2000	0.010560	0.025539
5000	0.006787	0.016018
10000	0.001574	0.013654

Table 2: *Kolmogorov distance and Anderson & Darling statistic for the estimates given in Table 7 (sampling distribution: maximum likelihood estimate for Bayer).*

sampling and the estimated distribution. For a better comparison we provide the Kolmogorov distance and the Anderson & Darling statistic in Table 2.

The fit of the tails becomes bad for sample sizes smaller than 150. From these results we obtain the rule of thumb that more than 150 observations are necessary for an acceptable fit to the tails.

6 Estimation with Different Metrics

In this section we apply different estimation methods by replacing the log-likelihood function by other metrics. The aim of these different approaches to the estimation is to investigate the possible improvement of the fit to the tails of the distribution. This may help for the modelling of the probability of extreme events. We estimate parameters for the GH, NIG and hyperbolic distributions using the metrics given in Section 4.

Is it useful to use different metrics for the estimation of return distributions? To answer this question we compare the empirical skewness and kurtosis with those values of the estimated distributions. The exact values of the skewness and kurtosis for a specified generalized hyperbolic distribution can be computed by the formulas given in Barndorff-Nielsen/Blæsild (1981). Both values are complicated expressions of Bessel functions. The results are given in Table 4.

Clearly generalized hyperbolic distributions provide a better fit to the empirical observed skewness and kurtosis than the normal distribution. But this depends on the method used to estimate the parameters.

The results given in Table 4 show that the Anderson & Darling statistic and the Kolmogorov distance are less useful for the estimation than the L^p-norms or the maximum likelihood approach. On the one hand the kurtosis of the estimated generalized hyperbolic distributions is always closer to the empirical kurtosis. On the other hand the estimated generalized hyperbolic

λ	α	β	δ	μ	value	χ	ξ	ρ	ζ
Maximum Likelihood									
-1.0024	39.6	4.14	0.0118	-0.000158	4878.00	0.086	0.827	0.104	0.463
NIG	59.4	4.64	0.0094	-0.000226	4877.62	0.063	0.802	0.078	0.556
HYP	114.8	3.35	0.0000	-0.000000	4872.25	0.029	1.000	0.029	0.000
Minimal Kolmogorov Distance									
-0.5002	63.8	3.81	0.0097	-0.000211	0.013469	0.047	0.786	0.060	0.620
NIG	63.8	3.81	0.0097	-0.000211	0.013470	0.047	0.786	0.060	0.621
HYP	116.8	4.60	0.0000	-0.000211	0.014424	0.039	1.000	0.039	0.000
Minimal Anderson & Darling Statistic									
-0.7162	39.6	4.00	0.0117	-0.000158	0.10	0.083	0.827	0.101	0.463
NIG	48.5	4.05	0.0118	-0.000158	0.13	0.067	0.799	0.084	0.568
HYP	80.5	2.98	0.0000	-0.000162	0.20	0.037	1.000	0.037	0.000
Minimal L^1-Distance									
0.0590	85.5	5.62	0.0073	-0.000282	0.000352	0.052	0.786	0.066	0.620
NIG	64.9	3.79	0.0098	-0.000064	0.000337	0.046	0.782	0.058	0.636
HYP	116.5	6.12	0.0000	-0.000328	0.000407	0.053	1.000	0.053	0.000
Minimal L^2-Distance									
0.4900	102.7	7.24	0.0052	-0.000459	0.00111	0.057	0.807	0.070	0.536
NIG	64.2	6.48	0.0098	-0.000382	0.00119	0.079	0.785	0.101	0.623
HYP	122.7	7.68	0.0022	-0.000503	0.00114	0.056	0.887	0.063	0.270

Table 3: *Estimation of the GH, NIG and hyperbolic distributions for the Deutsche Bank returns with different metrics.*

distributions are sometimes skewed in the other direction than the empirical distribution. Similar results are obtained for other stock data sets. In general the Anderson & Darling statistic and the Kolmogorov distance yield estimates for which skewness and kurtosis deviates in an irregular pattern from the empirical values. The estimates with L^p-norms are closer to the empirical kurtosis, but the estimation of the skewness is rather poor. Regarding also the other data sets, we obtain the best fits to the empirial skewness and kurtosis with the maximum likelihood approach. Therefore it is not favourable to replace the ML approach.

7 Value-at-Risk

A good fit of the heavy tails is also important for the estimation of the *Value-at-Risk* (VaR). The motivation for invention of the concept of Value-at-Risk was the necessity to quantify the risk for complex portfolios in a simple way. The VaR to a given *level of probability* α is defined as the maximal loss inherent to a portfolio position over a future *holding period* which is exceeded only with a probability of α. The level of probability is typically chosen as 1% or 5% and should not

Metric	Distribution	Skewness	Kurtosis	Skewness	Kurtosis
		Deutsche Bank		Bay.Hyp.u.Wechselbank	
Empirical		-0.519	10.872	-1.220	15.919
Normal		0.0	0.0	0.0	0.0
Maximum Likelihood	GH	0.378	7.492	0.291	10.413
Maximum Likelihood	NIG	0.314	5.529	0.178	4.490
Maximum Likelihood	HYP	0.123	3.010	0.071	3.003
Kolmogorov Distance	GH	0.227	4.906	-0.793	1.007
Kolmogorov Distance	NIG	0.227	4.903	-0.002	0.020
Kolmogorov Distance	HYP	0.166	3.018	-0.010	2.708
Anderson & Darling	GH	0.419	7.233	0.058	3.211
Anderson & Darling	NIG	0.332	5.427	-1.141	9.068
Anderson & Darling	HYP	0.156	3.016	0.043	2.579
L^1-Distance	GH	0.261	3.887	0.238	4.438
L^1-Distance	NIG	0.219	4.782	0.237	4.252
L^1-Distance	HYP	0.222	3.032	0.249	2.563
L^2-Distance	GH	0.281	3.315	0.323	4.087
L^2-Distance	NIG	0.383	5.010	0.320	4.024
L^2-Distance	HYP	0.246	2.764	0.227	3.034

Table 4: *Comparison of the directly estimated skewness and kurtosis with the skewness and kurtosis calculated from the estimations for GH, NIG and hyperbolic distributions with different metrics (Deutsche Bank and Bay.Hyp.u.Wechselbank returns).*

be confused with a confidence level. We are looking at the whole interval of levels of probability. This approach corresponds to the multivariate approach in Davé/Stahl (1997). We analyze the VaR for portfolios with linear risk, i.e. portfolios consisting of only one stock or index. The results of the VaR-estimation for the GH, NIG and hyperbolic distribution are given in Figure 2.

Obviously the class of generalized hyperbolic distributions and its sub-classes provide better fits to the empirical VaR, especially for small levels of probability, than the normal distribution.

The analysis of VaR for linear positions is also useful as a visualisation of the fitting behaviour in the tails of a distribution. From a mathematical point of view, VaR is in this case similar to the well-known qq-plots.

8 Conclusion

In this study we developed an algorithm to estimate parameters for the class of generalized hyperbolic distributions which includes the hyperbolic and the normal inverse Gaussian distri-

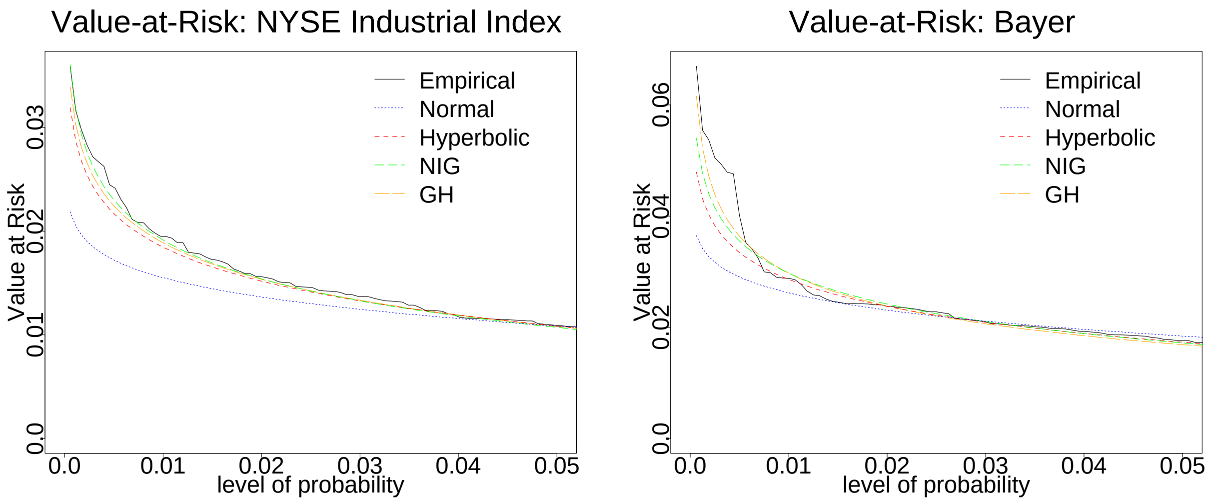

Figure 2: *VaR of a portfolio with linear risk and the value of one currency unit (US-$ or Deutsche Mark). The exposure period is one trading day. We compare the empirical VaR at different levels of probability to the estimated VaR using GH, NIG, hyperbolic and normal distributions.*

bution as special cases. We compared the results of the estimations for financial return data sets. In general, generalized hyperbolic distributions and their sub-classes provide better fits to the data than the normal distribution. As expected, the best fits are obtained for the generalized hyperbolic distributions followed by the NIG and the hyperbolic distributions. It is worth to mention that GH distributions lead to overfitting and that the estimation is computationally demanding. The hyperbolic distribution provides an acceptable tradeoff between the accuracy of the fit and and the necessary numerical effort.

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10 Tables

λ	α	β	δ	μ	LogLH	χ	ξ	ρ	ζ
NYSE Composite Index									
0.8357	214.4	-6.17	0.0022	0.000666	6399.07	-0.024	0.826	-0.029	0.466
NIG	136.6	-8.95	0.0059	0.000791	6397.57	-0.049	0.743	-0.066	0.811
HYP	211.6	-4.40	0.0000	0.000597	6396.03	-0.021	1.000	-0.021	0.000
NYSE Finance Index									
0.0680	152.4	-4.51	0.0062	0.000746	6070.17	-0.021	0.718	-0.030	0.939
NIG	125.5	-4.22	0.0078	0.000730	6070.05	-0.024	0.711	-0.034	0.977
HYP	174.7	-2.89	0.0000	0.000657	6061.61	-0.017	1.000	-0.017	0.000
NYSE Industrial Index									
0.2678	178.8	-8.81	0.0044	0.000843	6322.48	-0.037	0.749	-0.049	0.784
NIG	135.6	-9.36	0.0064	0.000872	6322.02	-0.051	0.733	-0.069	0.863
HYP	222.3	-8.74	0.0022	0.000834	6321.88	-0.032	0.817	-0.039	0.499
NYSE Transport Index									
-2.3017	6.9	6.89	0.0156	-0.000260	5729.41	1.000	1.000	1.000	0.000
NIG	109.4	7.66	0.0099	-0.000311	5725.87	0.049	0.692	0.070	1.086
HYP	169.1	8.10	0.0049	-0.000338	5721.87	0.035	0.738	0.048	0.835
NYSE Utility Index									
-1.1379	144.2	-9.15	0.0088	0.000521	6380.96	-0.042	0.664	-0.063	1.268
NIG*	180.1	-9.17	0.0075	0.000521	6380.88	-0.033	0.652	-0.051	1.354
HYP*	257.2	-9.44	0.0041	0.000529	6380.16	-0.026	0.698	-0.037	1.050

Table 5: *Maximum likelihood estimation of the parameters for generalized hyperbolic, NIG and hyperbolic distributions for New York Stock Exchange Indices from January 2, 1990 to November 29, 1996.*

λ	α	β	δ	μ	LogLH	χ	ξ	ρ	ζ
Allianz–Holding									
-1.8015	4.0	4.04	0.0175	-0.000172	4749.17	1.000	1.000	1.000	0.000
NIG	60.6	6.17	0.0110	-0.000501	4740.82	0.079	0.775	0.102	0.665
HYP	112.3	9.06	0.0033	-0.000948	4729.27	0.069	0.856	0.081	0.366
BASF									
-1.9594	3.8	3.09	0.0157	-0.000112	5030.18	0.793	0.983	0.807	0.036
NIG	82.6	4.55	0.0102	-0.000275	5025.99	0.041	0.738	0.055	0.838
HYP	140.2	5.90	0.0041	-0.000419	5019.47	0.034	0.796	0.042	0.576
Bayer									
-1.7882	21.3	2.67	0.0153	-0.000004	5003.69	0.109	0.869	0.125	0.323
NIG	81.6	3.69	0.0103	-0.000123	5001.54	0.033	0.737	0.045	0.843
HYP	139.0	5.35	0.0044	-0.000311	4996.00	0.030	0.789	0.039	0.608
Bay.Hyp.u.Wechselbank									
-1.5909	17.9	2.19	0.0157	-0.000072	4815.05	0.108	0.884	0.122	0.278
NIG	63.8	3.12	0.0106	-0.000211	4813.37	0.038	0.773	0.049	0.674
HYP	118.5	4.03	0.0035	-0.000330	4806.13	0.029	0.840	0.034	0.418
BMW									
-1.6630	9.0	2.73	0.0161	0.000048	4797.13	0.283	0.937	0.302	0.138
NIG	61.1	3.09	0.0105	0.000012	4793.19	0.039	0.780	0.051	0.643
HYP	115.0	3.78	0.0030	-0.000074	4783.72	0.028	0.862	0.033	0.345
Daimler Benz									
-1.6801	13.0	3.93	0.0182	-0.000539	4625.48	0.272	0.903	0.301	0.227
NIG	57.6	5.09	0.0120	-0.000748	4623.24	0.068	0.769	0.088	0.691
HYP	105.2	6.58	0.0039	-0.000999	4616.28	0.053	0.843	0.063	0.406
Deutsche Bank									
-1.0024	39.6	4.14	0.0118	-0.000158	4878.00	0.086	0.827	0.104	0.463
NIG	59.4	4.64	0.0094	-0.000226	4877.62	0.063	0.802	0.078	0.556
HYP	116.2	5.31	0.0009	-0.000290	4872.20	0.043	0.951	0.046	0.106
Lufthansa									
-0.5919	45.9	4.59	0.0177	-0.001271	4172.28	0.074	0.744	0.100	0.808
NIG	48.0	4.63	0.0170	-0.001283	4172.27	0.072	0.743	0.096	0.813
HYP	79.6	5.10	0.0050	-0.001415	4171.03	0.054	0.845	0.064	0.400
Siemens									
-1.8856	3.1	3.12	0.0164	-0.000002	4914.73	1.000	1.000	1.000	0.000
NIG	74.7	4.76	0.0107	-0.000188	4908.68	0.047	0.745	0.064	0.800
HYP	131.8	5.52	0.0049	-0.000266	4898.64	0.033	0.780	0.042	0.644

Table 6: *Maximum likelihood estimation of the parameters for generalized hyperbolic, NIG and hyperbolic distributions for German stocks from January 1988 to May 1994.*

λ	α	β	δ	μ	LogLH	χ	ξ	ρ	ζ
Sampling distribution (ML-estimation for Bayer)									
-1.7882	21.3	2.67	0.0153	-0.000004		0.109	0.869	0.125	0.323
Estimated parameters for n =10000									
-1.8012	3.0	2.95	0.0151	-0.000038	31189.29	1.000	1.000	1.000	0.000
NIG	71.4	3.62	0.0097	-0.000100	31149.38	0.039	0.770	0.051	0.689
HYP	121.4	2.42	0.0000	0.000062	31053.33	0.020	1.000	0.020	0.000
Estimated parameters for n =2000									
-1.6079	2.3	2.33	0.0135	0.000078	6282.07	1.000	1.000	1.000	0.000
NIG	60.1	3.32	0.0085	-0.000009	6263.12	0.045	0.814	0.055	0.508
HYP	122.6	5.06	0.0000	-0.000214	6228.70	0.041	1.000	0.041	0.000
Estimated parameters for n =1000									
-1.3570	36.7	-2.59	0.0123	0.000461	3185.43	-0.059	0.831	-0.071	0.448
NIG	76.2	-3.16	0.0089	0.000521	3184.83	-0.032	0.771	-0.041	0.680
HYP	130.8	-3.56	0.0000	0.000566	3180.12	-0.027	1.000	-0.027	0.000
Estimated parameters for n =500									
-1.2375	23.4	-2.62	0.0113	0.000399	1574.76	-0.100	0.890	-0.112	0.262
NIG	60.2	-3.61	0.0082	0.000515	1573.87	-0.049	0.819	-0.060	0.490
HYP	125.4	-4.25	0.0000	0.000566	1568.80	-0.034	1.000	-0.034	0.000
Estimated parameters for n =350									
-1.4060	12.7	1.98	0.0123	0.000320	1098.07	0.145	0.931	0.156	0.154
NIG	60.0	2.09	0.0084	0.000325	1096.90	0.028	0.816	0.035	0.501
HYP	123.4	2.36	0.0000	0.000306	1092.72	0.019	1.000	0.019	0.000
Estimated parameters for n =200									
-1.7709	16.6	8.09	0.0147	-0.000046	629.24	0.443	0.908	0.488	0.212
NIG	77.2	8.94	0.0096	-0.000118	628.86	0.088	0.759	0.116	0.736
HYP	126.6	10.56	0.0000	-0.000327	628.10	0.083	1.000	0.083	0.000
Estimated parameters for n =150									
-1.6539	14.3	6.04	0.0147	-0.000605	464.14	0.387	0.917	0.422	0.190
NIG	68.0	6.44	0.0096	-0.000624	463.80	0.074	0.778	0.095	0.652
HYP	119.1	2.86	0.0000	-0.000111	462.94	0.024	1.000	0.024	0.000
Estimated parameters for n =100									
0.6042	108.7	-11.68	0.0033	0.001483	312.97	-0.092	0.857	-0.107	0.361
NIG	67.3	-13.02	0.0087	0.001687	312.67	-0.154	0.797	-0.193	0.576
HYP	124.9	-6.61	0.0000	0.000816	313.12	-0.053	1.000	-0.053	0.000
Estimated parameters for n =50									
0.1545	109.4	4.80	0.0087	0.000808	155.27	0.031	0.716	0.044	0.949
NIG	86.3	4.14	0.0112	0.000888	155.25	0.034	0.713	0.048	0.966
HYP	120.3	6.48	0.0000	0.000528	154.69	0.054	1.000	0.054	0.000

Table 7: *Estimations for data sets with sample size n (sampling distribution: maximum likelihood estimate of the generalized hyperbolic distribution for Bayer returns).*