Power of surrogate data testing with respect to nonstationarity

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Surrogate data testing is a method frequently applied to evaluate the results of nonlinear time series analysis. Since the null hypothesis tested against is a linear, Gaussian, stationary stochastic process a positive outcome may not only result from an underlying nonlinear or even chaotic system, but also from, e.g., a nonstationary linear one. We investigate the power of the test against nonstationarity. [S1063-651X(98)14610-X]

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I. INTRODUCTION

The field of nonlinear dynamics introduced the fascinating idea that an apparently random behavior of a time series might have been generated by a low-dimensional deterministic system [1]. Based on the notions of chaos theory, different algorithms have been invented to infer if an observed time series is a realization of a chaotic system, e.g., the estimation of the largest Lyapunov exponent [2], the correlation dimension [3], and nonlinear prediction [4]. There is hope to gain deeper insights into complex systems like those from biology and physiology by applying these methods.

However, the application of these methods to a finite, often noisy set of measured data is not straightforward, see, e.g., [5–9], and references therein. For example, in order to claim a finite, fractal correlation dimension, a scaling region of sufficient length has to be established. Determining this scaling region by eye or some algorithm may lead to erroneous evidence of chaotic behavior. In order to evaluate the analysis, it has become popular to apply the method of surrogate data [5]. Therefore data are generated which have the same linear statistical properties as the original data but not the possible nonlinear ones. For many realizations of these data, the same algorithm as to the original data is applied. A significant difference between the distribution of the nonlinear feature for the surrogate data and the original data is taken as an indication that the process underlying the original data is deterministic [10], nonlinear [11–13], or even chaotic [14–16].

The explicit null hypothesis of surrogate data testing for linearity is that the data were generated by a linear, stochastic, Gaussian stationary process, including a possible invertible nonlinear observation function. Thus a rejection of this hypothesis does not necessarily mean that the data come from a chaotic, i.e., some kind of stationary, nonlinear deterministic, process. They might also originate from a nonlinear stochastic or even simply from a linear, stochastic, nonstationary process. In this paper we investigate the power of surrogate data testing against nonstationarity. As a nonlinear feature we use the correlation dimension. The behavior of correlation dimension estimates has been investigated for the $1/f^{\alpha}$, $\alpha \geqslant 1$ type of linear nonstationarity [17,18]. For physiological data, such $1/f$ behavior has been observed in heart rate [19]. Often, physiological data are characterized by some kind of oscillatory behavior like EEG, hormone secretion, breathing, or tremor. For such data, types of nonstationarity introducing some time dependency of the oscillating dynamics, e.g., a modulation of frequency or amplitude, seems to be a natural violation of the null hypothesis.

If the process is linear and the time dependency of the parameters, and thus, the autocovariance function is periodic in time, these processes are called cyclostationary [20]. Many other types of nonstationarity in oscillatory processes are imaginable. We choose cyclostationary processes because they allow a simple way to find a parametric violation of the null hypothesis. Formally, these processes can be expressed as higher-dimensional autonomous nonlinear stochastic processes. A special version of surrogate data testing acting on segments of the data has been suggested to analyze such data [21].

In the next section we informally discuss the class of cyclostationary processes and introduce the two specific examples we use in Sec. III to investigate the power of surrogate data testing with respect to these types of nonstationarity.

II. CYCLOSTATIONARY PROCESSES

The parameters a_i and σ^2 of a linear stochastic autoregressive (AR) process $x(t)$:

$$x(t)=\sum_{i=1}^{p} a_1 x(t-p)+\epsilon(t), \quad \epsilon(t)\sim\mathcal{N}(0,\sigma^2) \tag{1}$$

determine the autocovariance function $R(\tau)$:

$$R(\tau)=\langle x(t)x(t+\tau)\rangle. \tag{2}$$

The spectrum $S(\omega)$ is given as Fourier transform of the autocovariance function:

$$S(\omega)=\sum e^{-i\omega\tau}R(\tau). \tag{3}$$

A possible first step to nonstationarity is to define a time dependent spectrum $S(t,\omega)$ and, correspondingly, a time dependent autocovariance function $R(t,\tau)$:

$$R(t,\tau)=\langle x(t)x(t+\tau)\rangle. \tag{4}$$

A cyclostationary process of periodicity L is defined by

$$R(t,\tau)=R(t+L,\tau). \tag{5}$$

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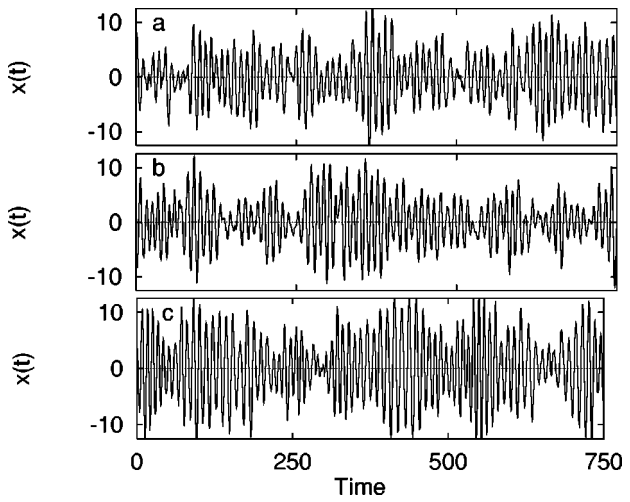

FIG. 1. Realizations of the processes investigated. (a) AR2 process satisfying the null hypothesis. (b) Amplitude modulated process with modulation depth of 0.3. (c) Period modulated process, relative amplitude of modulation is 15%.

For the AR process of Eq. (1) this means that the parameters a_i and σ^2 may change periodically.

As a process satisfying the null hypothesis of surrogate data testing for linearity, we chose an autoregressive process of order 2:

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,\sigma^2). \tag{6}$$

In terms of physics, AR processes can be interpreted as a combination of linear relaxators and linear damped oscillators driven by noise. For an AR process of order 2 which describes a damped oscillator, the parameters are related to the relaxation time τ and period T by

$$a_1 = 2\cos(2\pi/T)\exp(-1/\tau), \tag{7}$$

$$a_2 = -\exp(-2/\tau). \tag{8}$$

The variance of the process $\mathrm{Var}(x_t)$ is given by

$$\mathrm{Var}(x_t) = \frac{\sigma^2}{1-a_1^2-a_2^2-2a_1^2a_2/(1-a_2)}. \tag{9}$$

We choose an AR2 process with $T=10$, $\tau=50$, and $\sigma=1$ as process $x_0(t)$ that satisfies the null hypothesis. Figure 1(a) displays a realization of this process. The oscillatory behavior with a mean period of ten time steps is clearly visible as well as the natural variability of period and amplitude. Figure 2 (solid line) shows the estimated spectrum of the process. The spectrum was estimated by averaging 100 periodograms, i.e., the squared absolute value of the Fourier transform of the data. A broad peak, typical for a stochastically driven linear damped oscillator can be seen. Based on Eqs. (7)–(9) we now introduce two parametrized violations of this stationary, linear, stochastic process in order to investigate the power of surrogate data testing with respect to nonstationarity.

For the first violation of stationarity in the frame of cyclostationary processes, we choose a simple amplitude

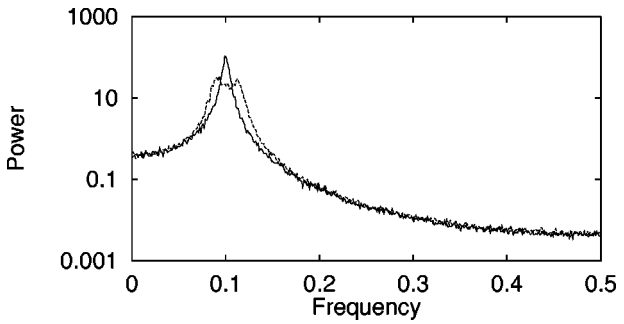

FIG. 2. Estimated spectra of the processes shown in Fig. 1. The spectra corresponding to Figs. 1(a) and 1(b) are not distinguishable (solid line). Period modulated process (dashed line).

modulation, corresponding by Eq. (9) to a periodicity of the variance of the driving noise. Based on the stationary AR2 process $x_0(t)$, the amplitude modulated process $x_{\mathrm{amp}}(t)$ is given by

$$x_{\mathrm{amp}}(t) = [1 + \mathcal{M}_{\mathrm{amp}} \sin(2\pi/T_{\mathrm{mod}} t)] x_0(t). \tag{10}$$

$\mathcal{M}_{\mathrm{amp}}$, the modulation depth, parametrizes the violation of the null hypothesis. T_{mod} determines the modulation period. Figure 1(b) displays a realization of this process with $T_{\mathrm{mod}} = 250$ and $\mathcal{M}_{\mathrm{amp}} = 0.3$ for three periods of the modulation. Compared to Fig. 1(a), the nonstationarity is hardly visible. Due to the long modulation period compared to the period of the process, its spectrum is not distinguishable from that of the stationary process in Fig. 2.

For the second violation of stationarity, we chose a modulation of the period T of the AR2 process with period T_{mod} and amplitude $\mathcal{M}_T$ around the mean period $T_{\mathrm{mean}} = 10$. This leads to a time dependency of the parameter a_1 of the AR2 process:

$$T(t) = T_{\mathrm{mean}} + \mathcal{M}_T \sin(2\pi/T_{\mathrm{mod}} t), \tag{11}$$

$$a_1(t) = 2\cos[2\pi/T(t)]\exp(-1/\tau). \tag{12}$$

$\mathcal{M}_T$ parametrizes the violation of the null hypothesis. According to Eq. (9), the time dependency of $a_1(t)$ causes a time dependency of the variance of the process. The effect of a changing variance is already covered by the first process, Eq. (10). To investigate only the effect of a changing period of the process here, we use Eq. (9) to adjust the variance $\sigma^2(t)$ of the driving noise such that the variance of the process is constant:

$$\sigma^2(t) = \frac{\sigma^2}{1-a_1^2-a_2^2-2a_1^2a_2/(1-a_2)} \tag{13}$$

$$\times \left(1 - a_1(t)^2 - a_2^2 - \frac{2a_1(t)^2 a_2}{1-a_2}\right), \tag{14}$$

where a_1 and σ^2 denote the parameters of the process $x_0(t)$ satisfying the null hypothesis. Figure 1(c) displays a realization of this process with $T_{\mathrm{mod}} = 250$ and $\mathcal{M}_T = 1.5$. Again, compared to Fig. 1(a), the nonstationarity is hardly visible. Figure 2 (dashed line) shows the estimated spectrum of the

process. The spectrum shows two peaks at the corresponding frequencies due to the specific type of modulation chosen.

III. POWER OF THE TEST

As a nonlinear feature to investigate the power of surrogate data testing against the two violations of stationarity we use the correlation dimension. The phase space is reconstructed by delay embedding. The delay is chosen equal to the lag at which the autocorrelation function first crosses zero.

The correlation dimension D_2 is defined by

$$D_2 = \lim_{r \to 0} \frac{d \ln C(r)}{d \ln r}, \tag{15}$$

where $C(r)$, the correlation integral, is given by

$$C(r) = \text{const} \sum_{i=1}^{N-\mu} \sum_{j=i+\mu}^{N} \Theta(r - |\vec{x}(i) - \vec{x}(j)|), \tag{16}$$

including the Theiler correction μ [22] which we chose equal to the mean period, i.e., ten time steps. The canonical procedure to establish a finite correlation dimension is to show the existence of a scaling region for small r where Eq. (15) holds and stays constant for a high enough embedding dimensions. For all processes investigated here, the true correlation dimension is infinity. Following the idea of surrogate data testing, we fix an algorithm to obtain a finite value from the correlation integral and look for differences to the original data. Therefore we apply Theiler and Lookman's ''rule of five'' chord estimator [23] and chose their R_0 equal to the standard deviation of the data. For such a large R_0 we do not examine the small scale behavior of Eq. (15) anymore. We are aware that we should not call this quantity correlation dimension anymore. It has been termed ''dimensional complexity'' [24].

The surrogate data are produced by the Fourier transform (FT) algorithm [5]. For each degree of violation of the null hypothesis 50 independent surrogate data sets of length 8192 were generated. Denoting the ''correlation dimension'' of the original data by f, the mean of the distribution of this feature for the surrogate data by μ_{surr}, and its variance by σ^2_{surr}, the result is displayed as

$$z = \frac{|f - \mu_{\text{surr}}|}{\sigma_{\text{surr}}}. \tag{17}$$

It was confirmed that the distribution of the feature is sufficiently well described by a Gaussian distribution. Thus z can be related to a confidence interval, since for 50 realizations the t distribution of $(f - \mu_{\text{surr}})/\sigma_{\text{surr}}$ is well approximated by a Gaussian distribution and $z = 1.96$ corresponds to the 5% level of significance.

In general, in power of the test investigations a procedure different from that outlined above is chosen. For a certain significance level, e.g., 5%, and different degrees of violation of the null hypothesis, numerous realizations, e.g., 1000, of the process are generated and the fraction of rejected null hypotheses is reported. Due to the high computational burden for calculating the correlation integral, this procedure is not feasible here. The above procedure has the drawback that the results depend on the single realization that is used as a basis for the surrogates. We repeated the analysis reported below for independent realizations and found no qualitative differences for different realizations.

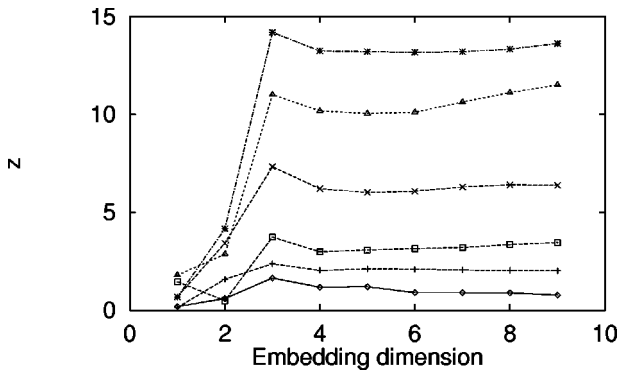

FIG. 3. Results of the simulation study for the amplitude modulated process. Shown is z in dependence on the embedding dimension for different degrees $\mathcal{M}_{\text{amp}}$ of violation ($\diamond = 0$, $+ = 0.1$, $\square = 0.2$, $\times = 0.3$, $\triangle = 0.4$, $* = 0.5$).

For the first violation of the null hypothesis, we increase $\mathcal{M}_{\text{amp}}$ in Eq. (10) from zero, i.e., no violation, to 0.5 in steps of 0.1. The distribution of these data is not Gaussian for $\mathcal{M}_{\text{amp}} > 0$. Thus, the amplitude adjusted surrogate data algorithm [5] was applied. The deviation from Gaussianity is weak for the range of violations chosen. We also applied the algorithm without amplitude adjustment and did not find significantly different results.

Figure 3 displays the result of the simulation study. In dependence on the embedding dimension, z is displayed for different degrees of violation of the null hypothesis. As expected, without any violation, the z values stay within the 2σ region given by $z \leqslant 1.96$. A modulation depth $\mathcal{M}_{\text{amp}}$ of 0.1 and 0.2 leads to results at the border of 5% significance. Starting from $\mathcal{M}_{\text{amp}} = 0.3$, see Fig. 1(b), the null hypothesis is clearly rejected at the 5% level of significance whenever the embedding dimension is large enough to reconstruct the second order process appropriately.

To investigate the effect of a variation in the period of the linear stochastic process, we increase $\mathcal{M}_T$ in Eqs. (11) and (12) from zero to three. The distribution of these data is

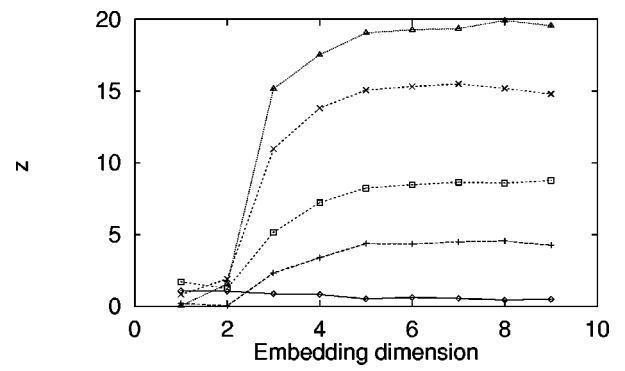

FIG. 4. Results of the simulation study for the period modulated process. Shown is z in dependence on the embedding dimension for different degrees $\mathcal{M}_T$ of violation ($\diamond = 0$, $+ = 1$, $\square = 1.5$, $\times = 2$, $\triangle = 3$).

Gaussian independent from the value of $\mathcal{M}_T$. Thus no amplitude adjustment was necessary. Again, the distribution of the feature is sufficiently well described by a Gaussian distribution. Figure 4 displays the result of the simulation study. For all degrees of violation, the violation is not detected when the embedding dimension is too small to unfold the dynamics in phase space. Otherwise, a modulation of the period of 15%, see Fig. 1(c), leads to a clear rejection of the null hypothesis at the 5% level of confidence.

IV. CONCLUSION

The simulation studies reported in this paper indicate that surrogate data testing for linear, stochastic, Gaussian stationary processes is powerful against a violation of the assumption of stationarity. Thus a significant result of the test does not necessarily indicate a nonlinear or even chaotic process underlying the data. It might simply be caused by a nonstationarity of the process.

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