Comments:

I will present work in progress done with several collaborators: Alfredo Gonzalez, Andrew Fleck and Alexey Rubtsov.

Trajectory Based Market Models with Operational Assumptions

Sebastian Ferrando, Department of Mathematics, Ryerson University, Toronto, Canada

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- We briefly outline how one can obtain a price interval for options in a trajectory based (non-probabilistic) setting.
- We point out about the broad generality of such trajectorial framework.
- We spend most of the talk describing a trajectorial operationally based market model and its construction.

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- We ask: is there a methodology that identifies market conditions implying a specific probability distribution for future asset's values?
- In other words: is there an objective method to justify the assumption of a definite probability model? Or, at least to identify when an assumed probability law is not being satisfied?
- The answer, it seems to me, is a clear NO.

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- It is difficult to judge the seal of approval for a specific model given by statistics. There are many judgement calls in the application of statistics, e.g. is model 1 better than model 2 (for example when models are not nested)? What changes in market conditions will invalidate the estimated parameter values? Should I assume some ambiguity on my probability distribution?

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- There are not many studies on the limitations of stochastic modelling in finance (and elsewhere). I refer to: Chameleons: *The Misuse of Theoretical Models in Finance and Economics* By Paul Pfleiderer March 2014 Working Paper No. 3020. Stanford University (*Revista de Economia Institucional*, **16**, 31, 23 – 60).

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- We return to this topic when we introduce operational models.

Trajectorial Setting

- A process $X = \{X_t\}$ can be thought as: $X : (\Omega, \mathcal{F} = \{\mathcal{F}_t\}) \to \mathbb{R}^{[0,T]}$. The process' measure P is used to pick up a set of trajectories $\mathcal{J} = X(\Omega')$ where Ω' is a set of full measure. There may be several measures Q with same null sets (i.e. equivalent measures).

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- We propose to start with a **set of trajectories** $\mathcal{J}$ and a **set of portfolios** $\mathcal{H}$ that provide a **Non Probabilistic (NP)** model $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ which is free of arbitrage in a trajectorial sense.

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- The theory is developed from first principles in discrete time (but the number of transactions can be unbounded and there are no cardinality restrictions)

Discrete Setting: Trajectories and Portfolios

Trajectory set:

$$\mathcal{J} = \mathcal{J}(s_0) = \{S = \{S_i\}_{i=0}^{\infty} : S_i \in \mathbb{R}\}, \; S_0 = s_0 \; \forall S$$

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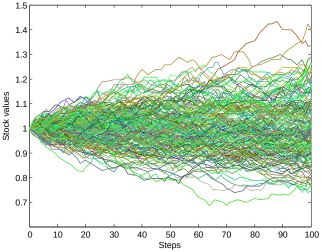

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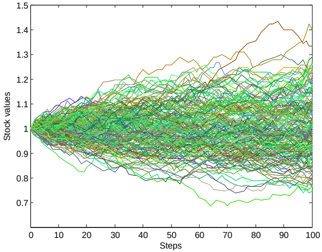

Discrete Setting: Portfolio Values

Portfolio value, just before trading at instance n, is given by:

$$V_H(n,S) = H_{n-1}(S)S_n + B_{n-1} = V_H(0,S) + \sum_{k=0}^{n-1} H_k(S)(S_{k+1} - S_k)$$

so we are imposing a self-financing condition on the portfolio $\Phi = \{(H_i, B_i)\}$ where B_i are the holdings in the bank account.

Conditional Sets of Trajectories

For $S \in \mathcal{J}$ and $j \geq 0$ define

$$\mathcal{J}_{(S,k)} = \{\hat{S} \in \mathcal{J} : \hat{S}_j = S_j,\ 0 \leq j \leq k\}. \tag{1}$$

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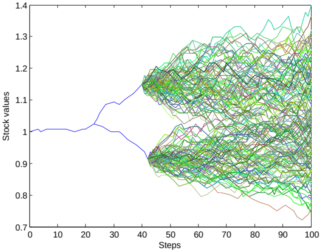

Conditional Up Down Property

$\mathcal{J}$ is said to satisfy the conditional up down property if for S and j fixed

$$\sup_{\hat{S} \in \mathcal{J}_{(S,j)}} (\hat{S}_{j+1} - S_j) > 0, \quad \text{and} \quad \inf_{\hat{S} \in \mathcal{J}_{(S,j)}} (\hat{S}_{j+1} - S_j) < 0, \tag{2}$$

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for any $j \geq 0$ and any $S \in \mathcal{J}$

No Arbitrage

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Theorem

Any discrete market $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ such that $\mathcal{J}$ satisfies the up down property is arbitrage-free.

Pricing in Trajectory Markets

- In the usual, risk neutral approach, the price of a European option $Y(T, w)$ is given by an expectation (consider interest rates $r = 0$)

$$\mathbb{E}_Q(Y(T, \cdot)). \tag{4}$$

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- In a trajectory based approach, it follows that one is required to use a minmax optimization.

Minimax Bounds

- Let $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ and Z a function defined on $\mathcal{J}$, define:

$$\overline{V}(Z_T) \equiv \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{J}} [Z(S) - \sum_{i=0}^{N(S)-1} H_i(S)\,(S_{i+1} - S_i)]. \text{ And } \underline{V}(Z) = -\overline{V}(-Z)$$

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Explanation: given a portfolio $\Phi \in \mathcal{H}$, with zero initial value, we look for the trajectory with the worst error for the final portfolio value. This worst error gives the needed shift (initial capital) to superhedge (for all trajectories). Then, we look for the best portfolio, the one where this shift (initial capital) is smallest.

0-Neutral And Pricing Interval

We will say that $\mathcal{M}$ is 0-neutral if:

$$\overline{V}(0) \equiv \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{J}} - \sum_{i=0}^{N(S)-1} H_i(S)\,(S_{i+1} - S_i) = 0. \tag{5}$$

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If $\mathcal{M}$ is 0-neutral then

$$\underline{V}(Z) \leq \overline{V}(Z)$$

Meaning of Model $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ for Pricing

- It can happen that the bounds are too wide as they are worst case based.

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- Hopefully, in our example, you will see why one can not make an a-priori judgement on the meaning of worst case.

Properties:

1. $\overline{V}(Z)$ and $\underline{V}(Z)$ are the minimum and maximum initial capitals required, respectively, to superhedge and underhedge uniformly in a trajectory based sense.

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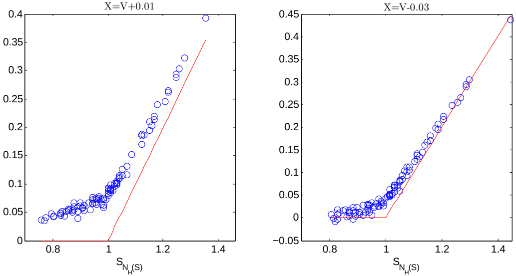
2. The superhedge and underhedge mentioned above are tight, namely, if a portfolio has initial value smaller that $\overline{V}(Z)$, there is at least one trajectory $x^* \in \mathcal{J}$ such that the portfolio is below the payoff at x^*. A dual statement also holds for $\underline{V}(Z)$.

Example



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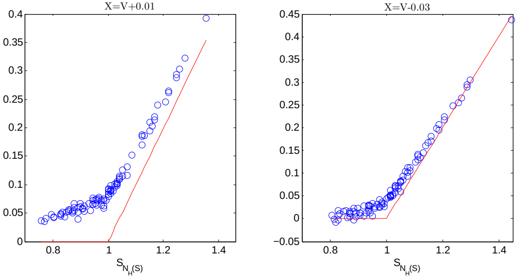

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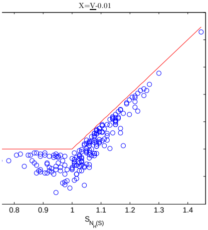
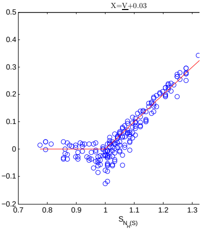

X=V-0.01
0.8
0.9
1
1.1
1.2
1.3
1.4
$S_{N_H}(S)$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2. It is this lack that precludes an analysis of changing market conditions and modeling assumptions.
3. As the asset unfolds how could we know if the BM assumptions are not being fulfilled?

General Features of Example

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1. One single risky asset and one riskless bank account. The setting and example are constructed to illustrate a general methodology.
2. The goal is to define models based on observable quantities that relate directly to a class of investors interested in gauging the price of an European option Z written on the asset.
3. Trajectories emerge as a result of constraints that correspond to how a class of investors sample a financial chart and how they rebalance their portfolios as a response to changes of their data summaries. We allow for all possible trajectories satisfying the said constraints (we call this a combinatorial definition).

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Hence, we deal with a class of investors that will react to chart changes of size δ and could observe chart values at a certain time resolution Δ and space resolution δ_0. From this perspective, these portfolio managers want to evaluate investment opportunities related to an European option. They want to compare market prices relative to their operational investment setup.

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Definition (dynamic sampling times)

Given $\delta_0 > 0$, a chart x and interval $[t_0, t_0 + T]$; a sequence of increasing dynamic sampled times is given by $r = r(x) = \{r_l\}_{l=0}^{L} \subseteq \Delta\ \mathbb{Z}$, $L = L(x)$, $r_0 = t_0$, satisfying:

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notice that $\delta_0 \leq |x(r_L) - x(r_{L-1})|$ may or may not hold. We also require the times r_l to be *tight*, namely: if $r_l < t < r_{l+1}, t \in \Delta\ \mathbb{Z}$, then $\delta_0 > |x(t) - x(r_l)|$.

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The times t_i are chosen in a *greedy* way, namely given $t_i = r_{l_i}$, t_{i+1} is the smallest element $r_{l_{i+1}}$ in $\{r_l\}$ satisfying the above equation if such element exists, otherwise $r_{l_{i+1}} = T$ and $N = N(x) = l_{i+1}$.

Implied Constraints

We can then write for $0 \leq i \leq N(x) - 1$:

$$\Delta_i x = \sum_{j=0}^{s_i - 1} (x(r_{l_i + j + 1}) - x(r_{l_i + j})) = \delta_0 \sum_{j=0}^{s_i - 1} p_j = m_i \quad \delta_0$$

so

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Also,

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Therefore

$$s_i\ \Delta \leq \sum_{l=l_i}^{l_i+s_i} (r_{l+1} - r_l) = q_i\ \Delta, \text{ i.e. } 1 \leq s_i \leq q_i \leq M_T.$$

Implied Constraints

Moreover, given that times t_i are chosen in a greedy fashion we also have for any $1 \leq R < s_i$

$$|\sum_{j=0}^{R-1} p_j| < Z,\ 1 \leq |p_j| < 2\ Z,\ 0 \leq j \leq s_i - 2,\ \ |p_{s_i-1}| \leq (|m_i| + Z).$$

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Given that $m_i\ \delta_0 \equiv x(t_{i+1}) - x(t_i)$, hence $|m_i|\delta_0 \geq \delta$ is guaranteed for $i < N(x) - 1$ and so

$$|m_i| \geq Z, \quad \text{for} \qquad 0 \leq\ i < N(x) - 1.$$

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$$|\sum_{j=0}^{R-1} p_j| < Z,\ 1 \leq |p_j| < 2\ Z,\ 0 \leq j \leq s_i - 2,\ \ |p_{s_i - 1}| \leq (|m_i| + Z).$$

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$$|m_i| \geq Z, \quad \text{for} \qquad 0 \leq\ i < N(x) - 1.$$

$$\Delta_i w \equiv w(x, t_{i+1}) - w(x, t_i) \equiv \sum_{j=0}^{s_i - 1} |x(r_{l_i + j + 1}) - x(r_{l_i + j})| \equiv$$

$$\delta_0\ \sum_{j=0}^{s_i - 1} |p_j| \equiv (j_{i+1} - j_i)\ \delta_0, \quad \text{so} \quad w(x, t_i) = \sum_{l=0}^{l_i - 1} |x(r_{l+1}) - x(r_l)|.$$

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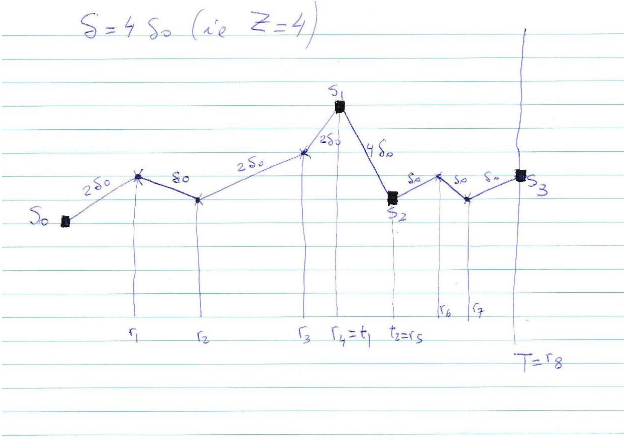
- p_j the possible number of δ_0 chart units in between two consecutive samples,
- $N(x)$ total number of portfolio rebalances in $[t_0, t_0 + T]$.

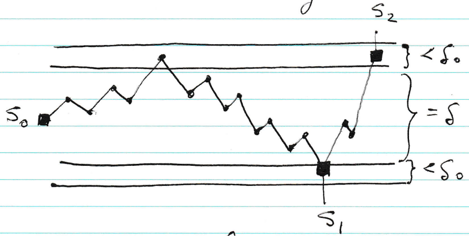
We will rely on $s_i \leq q_i$, that is we are taking an upperbound.

δ_0-Continuity, Jumps and In-Between Variation Changes



Extreme Case, Model Upcrossings

more dramatically



So we are observing upcrossings

Definition of Trajectory Set

Actually we work with vector valued trajectories i.e. $S_i \rightarrow \mathbf{S_i} \equiv (S_i, T_i, W_i)$, so we construct $\mathcal{S}$ with elements $\mathbf{S} \in \mathcal{S}$, $\mathbf{S} = \{\mathbf{S}_i\}$. We just define a set of trajectories satisfying the previous constraints (we do introduce, later, external, empirical constraints on parameters). In other words: (S_i, T_i, W_i) is defined by the obtained constraints obeyed by $(x(t_i), t_i, w(x, t_i))$.

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Given a triple (k_i, n_i, j_i) we allow for all possible $(k_{i+1}, n_{i+1}, j_{i+1})$ satisfying the previous constraints; for each such admissible triple we set:

$$\Delta_i S \equiv (S_{i+1} - S_i) = (k_{i+1} - k_i)\delta_0 = m_i \ \delta_0,$$

$$\Delta_i W \equiv (W_{i+1} - W_i) = (j_{i+1} - j_i)\delta_0 = \sum_{j=0}^{s_i - 1} |p_j|\delta_0,$$

$$\Delta_i T \equiv (T_{i+1} - T_i) = (n_{i+1} - n_i)\Delta = q_i \ \Delta, \text{ where } \quad m_i, \ p_j \in \mathbb{Z}.$$

$1 \leq |p_j|, \ 1 \leq q_i \leq M_T, \ 1 \leq s_i \leq q_i, \ s_i, q_i \in \mathbb{N}$

Definition of Trajectory Set

Moreover, define

$$N(\mathbf{S}) = i \quad \text{where} \;\; T_i = T, \tag{6}$$

we remark that $N(\mathbf{S})$ exists as $\Delta_i T \geq \Delta$ for any $i \geq 0$.
For the case $i < N(\mathbf{S}) - 1$ we also will require $|m_i| \geq \frac{\delta}{\delta_0} = Z$. Also, for any $1 \leq R < s_i$,

$$\sum_{j=0}^{s_i-1} p_j = m_i \,, \;\; |\sum_{j=0}^{R-1} p_j| < Z, \; |p_j| < 2\,Z, \; 0 \leq j \leq s_i - 2, \;\; |p_{s_i-1}| \leq (|m_i| + Z).$$

Empirical Restrictions Via Worst Case Estimated Parameters

- Let $N(x^*, [t, t+T])$ be the number of δ-moves. Let N_1 be the minimum of $N(x^*, [t, t+T])$ over historical data $x^*(t)$. Let N_2 be the analogous maximum.

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- Let $\mathcal{N}_E \equiv \{(m, q) = (\frac{x^*(t_{i+1}) - x^*(t_i)}{\delta_0}, \frac{t_{i+1} - t_i}{\Delta})$: collecting pairs over historical data $x^*(t)\}$

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- Similarly, for $0 \leq \rho \leq T$ define

$$V_\rho(x^*, [t, t+T]) \equiv \sum_{l=0}^{r_{l+1} \leq t+\rho} |x^*(r_{l+1}) - x^*(r_l)|,$$

Moreover, define

$$V_\rho^* \equiv \cup_{[t,t+T]\subseteq \mathcal{T}} \quad \{V_\rho(x^*, [t, t+T])\},$$

and require

$$W_{T_i} \in V^*_{T_i}. \tag{7}$$

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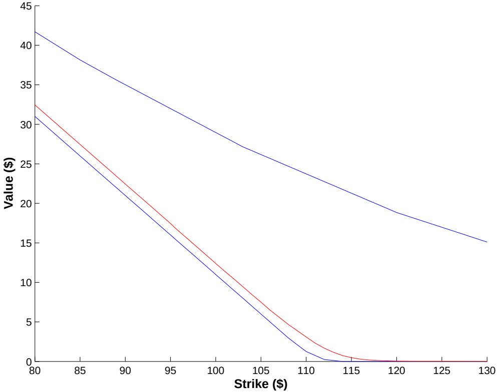
- A global constraint such as the above is what blocks a local extreme parameter (say (m_{max}, q)) of being available at all nodes (if that is the case there will be a trajectory with too large a value of variation). Therefore, global constraints, in this case, tame the effect of worst case estimation on the price option bounds.

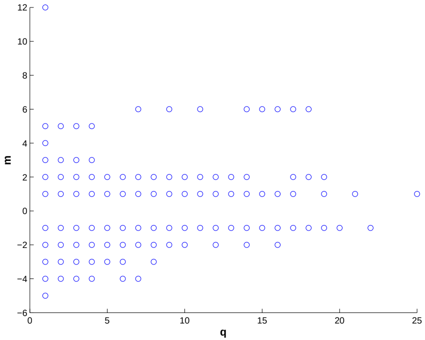
The two sources of data used for calibration and estimation are:

- A 6 month long stretch of hourly data ticks for Facebook Inc (FB). At the end of this period, call option ask prices were captured across a variety of strikes for comparison with an expiration of 9 days into the future.
- A 6 month long stretch of hourly ticks data for Biogen Inc (BIIB). At the end of this period, call option ask prices were captured across a variety of strikes for comparison with an expiration of 15 days into the future.

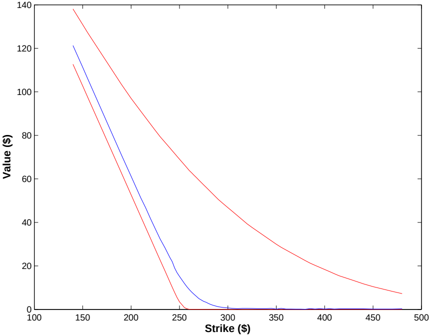
FB Price Bounds using observed conditional set



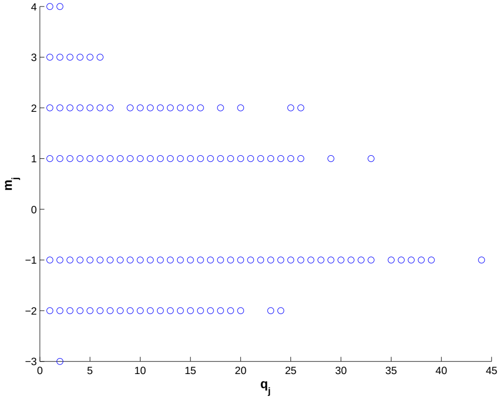
FB data (without using variation constraint).



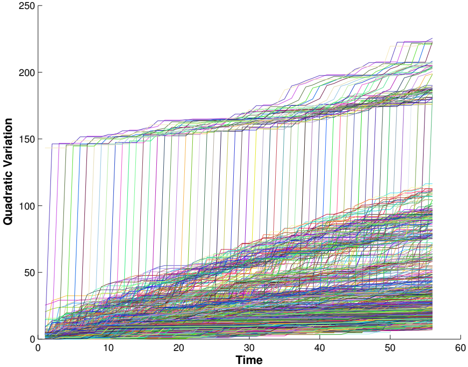
Price Bounds using observed conditional set BIIB Data



BIIB observed conditional set without variation constraint

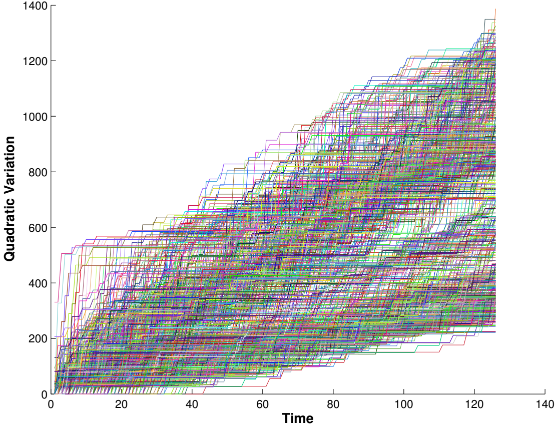


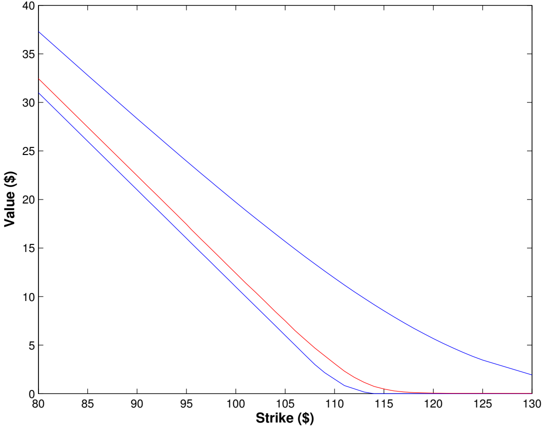
FB QV data



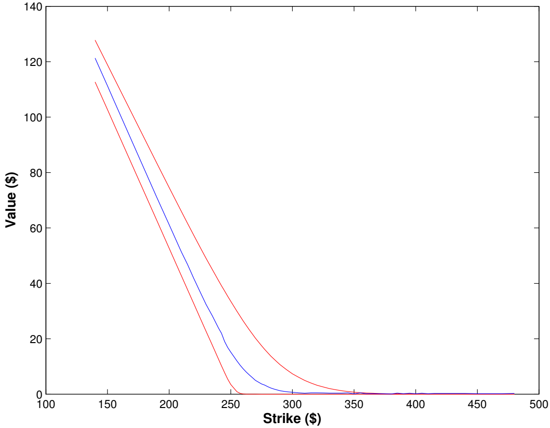
250
200
150
100
50
0
Quadratic Variation
0
10
20
30
40
50
60
Time

BIIB QV data


Quadratic Variation
Time
0
200
400
600
800
1000
1200
1400
0
20
40
60
80
100
120
140



Value ($)
Strike ($)
0
5
10
15
20
25
30
35
40
80
85
90
95
100
105
110
115
120
125
130



140
120
100
80
60
40
20
0
Value ($)
100
150
200
250
300
350
400
450
500
Strike ($)

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- One can trade risk and reward by removing some infrequent pairs (m, q) or by making N_1 larger or N_2 smaller or by restricting the set V^*_ρ.
- One can then bet on some rewards and on a risk that is worst case reliable.

Thank you!