I will present work in progress done with several collaborators: Alfredo Gonzalez, Andrew Fleck and Alexey Rubtsov.
Trajectory Based Market Models with Operational Assumptions

Sebastian Ferrando, Department of Mathematics, Ryerson University, Toronto, Canada
We briefly discuss reasons that suggest limitations with probabilistic modeling and in particular with *indiscriminated* stochastic modeling.
Summary:

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- We briefly outline how one can obtain a price interval for options in a trajectory based (non-probabilistic) setting.
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We point out about the broad generality of such trajectorial framework.

We spend most of the talk describing a trajectorial operationally based market model and its construction.
Non Justified Randomness

We ask: is there a methodology that identifies market conditions implying a specific probability distribution for future asset’s values?
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In other words: is there an objective method to justify the assumption of a definite probability model? Or, at least to identify when an assumed probability law is not being satisfied?

The answer, it seems to me, is a clear NO.
Statistical Justification

- The implicit justification of stochastic models is based on statistics.
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It is difficult to judge the seal of approval for a specific model given by statistics. There are many judgement calls in the application of statistics, e.g. is model 1 better than model 2 (for example when models are not nested)? What changes in market conditions will invalidate the estimated parameter values? Should I assume some ambiguity on my probability distribution?
Pletora of Models

- In practice models are proposed in multitude and with no objective criteria to discriminate among them.
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We return to this topic when we introduce operational models.
A General Trajectorial Framework

Trajectorial Setting

- A process $X = \{X_t\}$ can be thought as: $X : (\Omega, \mathcal{F} = \{\mathcal{F}_t\}) \rightarrow \mathbb{R}^{[0,T]}$. The process’ measure $P$ is used to pick up a set of trajectories $\mathcal{J} = X(\Omega')$ where $\Omega'$ is a set of full measure. There may be several measures $Q$ with same null sets (i.e. equivalent measures).
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- We propose to start with a set of trajectories $\mathcal{J}$ and a set of portfolios $\mathcal{H}$ that provide a Non Probabilistic (NP) model $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ which is free of arbitrage in a trajectorial sense.
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- The theory is developed from first principles in discrete time (but the number of transactions can be unbounded and there are no cardinality restrictions).
Discrete Setting: Trajectories and Portfolios

Trajectory set:

\[ \mathcal{J} = \mathcal{J}(s_0) = \{ S = \{ S_i \}_{i=0}^{\infty} : S_i \in \mathbb{R} \}, \ S_0 = s_0 \ \forall S \]
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Discrete Setting: Portfolio Values

Portfolio value, just before trading at instance $n$, is given by:

$$V_H(n, S) = H_{n-1}(S)S_n + B_{n-1} = V_H(0, S) + \sum_{k=0}^{n-1} H_k(S)(S_{k+1} - S_k)$$

so we are imposing a self-financing condition on the portfolio $\Phi = \{(H_i, B_i)\}$ where $B_i$ are the holdings in the bank account.
Conditional Sets of Trajectories

For $S \in \mathcal{J}$ and $j \geq 0$ define

$$\mathcal{J}(S, k) = \{ \hat{S} \in \mathcal{J} : \hat{S}_j = S_j, \; 0 \leq j \leq k \}.$$  \hfill (1)
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Conditional Up Down Property

\( \mathcal{J} \) is said to satisfy the **conditional up down property** if for \( S \) and \( j \) fixed

\[
\sup_{\hat{S} \in \mathcal{J}(S,j)} (\hat{S}_{j+1} - S_j) > 0, \quad \text{and} \quad \inf_{\hat{S} \in \mathcal{J}(S,j)} (\hat{S}_{j+1} - S_j) < 0, \quad (2)
\]

or:

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\sup_{\hat{S} \in \mathcal{J}(S,j)} (\hat{S}_{j+1} - S_j) = \inf_{\hat{S} \in \mathcal{J}(S,j)} (\hat{S}_{j+1} - S_j) = 0 \quad (3)
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for any \( j \geq 0 \) and any \( S \in \mathcal{J} \)
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No Arbitrage

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**Theorem**

Any discrete market $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ such that $\mathcal{J}$ satisfies the up down property is arbitrage-free.
In the usual, risk neutral approach, the price of a European option $Y(T, \omega)$ is given by an expectation (consider interest rates $r = 0$)

$$E_Q(Y(T, \cdot)).$$  \hspace{1cm} (4)

where $Q$ a chosen risk neutral measure (there are many in the incomplete case).
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(4)

where $Q$ a chosen risk neutral measure (there are many in the incomplete case).

In a trajectory based approach, it follows that one is required to use a minmax optimization.
Minimax Bounds

Let $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ and $Z$ a function defined on $\mathcal{J}$, define:

$$
\overline{V}(Z_T) \equiv \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{J}} \left[ Z(S) - \sum_{i=0}^{N(S)-1} H_i(S) (S_{i+1} - S_i) \right].
$$

And $\underline{V}(Z) = -\overline{V}(-Z)$. 

Explanation: given a portfolio $\Phi \in \mathcal{H}$, with zero initial value, we look for the trajectory with the worst error for the final portfolio value. This worst error gives the needed shift (initial capital) to superhedge (for all trajectories). Then, we look for the best portfolio, the one where this shift (initial capital) is smallest.
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And $V(Z) = -\bar{V}(-Z)$.

Explanation: given a portfolio $\Phi \in \mathcal{H}$, with zero initial value, we look for the trajectory with the worst error for the final portfolio value. This worst error gives the needed shift (initial capital) to superhedge (for all trajectories). Then, we look for the best portfolio, the one where this shift (initial capital) is smallest.
0-Neutral And Pricing Interval

We will say that $\mathcal{M}$ is 0-neutral if:

$$\overline{V}(0) \equiv \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{J}} \left( \sum_{i=0}^{N(S)-1} H_i(S) (S_{i+1} - S_i) \right) = 0. \quad (5)$$

**Proposition**

*If $\mathcal{M}$ is arbitrage free then $\mathcal{M}$ is 0-neutral.*
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\underline{V}(Z) \leq \overline{V}(Z)
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Meaning of Model $\mathcal{M} = \mathcal{J} \times \mathcal{H}$ for Pricing

- It can happen that the bounds are too wide as they are worst case based.
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- It can happen that the bounds are too wide as they are worst case based.
- Hopefully, in our example, you will see why one cannot make an a-priori judgement on the meaning of worst case.
Properties:

1. \( \overline{V}(Z) \) and \( \underline{V}(Z) \) are the minimum and maximum initial capitals required, respectively, to superhedge and underhedge uniformly in a trajectory based sense.
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2. The superhedge and underhedge mentioned above are tight, namely, if a portfolio has initial value smaller than $\overline{V}(Z)$, there is at least one trajectory $x^* \in \mathcal{J}$ such that the portfolio is below the payoff at $x^*$. A dual statement also holds for $\underline{V}(Z)$. 
Notice that (in principle) the approach permits to study the dependency of the superhedging gap as a function of the trajectories (i.e. which trajectories inflate, in a biased way for the seller, the price interval).

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Example
From our point of view a main issue with “out of the can” stochastic modeling is the lack of identification of the driving process (say BM) and observable market features.
Main Obstruction for Rigorous Model Falliiability Analysis

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2. It is this lack that precludes an analysis of changing market conditions and modeling assumptions.
Main Obstruction for Rigorous Model Falliability Analysis

1. From our point of view a main issue with “out of the can” stochastic modeling is the lack of identification of the driving process (say BM) and observable market features.

2. It is this lack that precludes an analysis of changing market conditions and modeling assumptions.

3. As the asset unfolds how could we know if the BM assumptions are not being fulfilled?
General Features of Example

1. One single risky asset and one riskless bank account. The setting and example are constructed to illustrate a general methodology.
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General Features of Example

1. One single risky asset and one riskless bank account. The setting and example are constructed to illustrate a general methodology.

2. The goal is to define models based on observable quantities that relate directly to a class of investors interested in gauging the price of an European option $Z$ written on the asset.

3. Trajectories emerge as a result of constraints that correspond to how a class of investors sample a financial chart and how they rebalance their portfolios as a response to changes of their data summaries. We allow for all possible trajectories satisfying the said constraints (we call this a combinatorial definition).
General Features of Example

For fixed $\delta > \delta_0 > 0$, $\Delta > 0$:

1. Charts $x$ are sampled at dynamic times $r_l$ according to $\delta_0$. 

2. Time intervals have a lower bound resolution $\Delta$ in the sense that $(r_{l+1} - r_{l}) \geq \Delta > 0$.

3. Times $t_i$, for the $i$-th portfolio rebalance, satisfy $t_i \in \{r_l\}$ and are given by an investor prescribed threshold $\delta$.

4. Sampled variation $w(x, t)$ is accumulated for the samples $x(r_l)$.

5. There is a dynamical number of portfolio rebalances $N(x)$ that take place in $[t_0, t_0 + T]$. Hence, we deal with a class of investors that will react to chart changes of size $\delta$ and could observe chart values at a certain time resolution $\Delta$ and space resolution $\delta_0$. From this perspective, these portfolio managers want to evaluate investment opportunities related to an European option. They want to compare market prices relative to their operational investment setup.
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Sampling Times

Assume

\[ x(t) \in \{ k\delta_0 : k \in \mathbb{Z} \}, \quad \delta = Z \delta_0, \quad Z \in \mathbb{N}. \]
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**Definition (dynamic sampling times)**

Given \( \delta_0 > 0 \), a chart \( x \) and interval \([t_0, t_0 + T]\); a sequence of increasing dynamic sampled times is given by \( r = r(x) = \{ r_l \}_{l=0}^L \subseteq \Delta \mathbb{Z}, \ L = L(x), \ r_0 = t_0 \), satisfying:

\[ \delta_0 \leq |x(r_{l+1}) - x(r_l)|, \quad 0 \leq l < L - 1, \quad r_L = t_0 + T, \]
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notice that \( \delta_0 \leq |x(r_L) - x(r_{L-1})| \) may or may not hold. We also require the times \( r_l \) to be tight, namely: if \( r_l < t < r_{l+1}, \ t \in \Delta \mathbb{Z}, \) then
\[ \delta_0 > |x(t) - x(r_l)|. \]
Rebalancing Rimes

Definition (dynamic rebalancing times)

Given $\delta > 0$, a chart $x$ and interval $[t_0, t_0 + T]$, a sequence of increasing dynamic rebalancing times is given by $t = t(x) = \{t_i\}_{i=0}^N \subseteq \{r_i\}_{i=0}^L$, $N = N(x)$, $t_0(x) = t_0$, satisfying:

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$$\delta \leq |x(t_{i+1}) - x(t_i)|, \quad 0 \leq i < N - 1, \quad t_N = t_0 + T,$$

The times $t_i$ are chosen in a greedy way, namely given $t_i = r_l$, $t_{i+1}$ is the smallest element $r_{l_{i+1}}$ in $\{r_l\}$ satisfying the above equation if such element exists, otherwise $r_{l_{i+1}} = T$ and $N = N(x) = l_{i+1}$. 
Implied Constraints

We can then write for $0 \leq i \leq N(x) - 1$:

$$\Delta_i x = \sum_{j=0}^{s_i-1} (x(r_{l_i+j+1}) - x(r_{l_i+j})) = \delta_0 \sum_{j=0}^{s_i-1} p_j = m_i \delta_0$$

So

$$\sum_{j=0}^{s_i-1} p_j = m_i.$$

where $t_i = r_{l_i} < \ldots < r_{l_i+s_i} = t_{i+1}$. 
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where $t_i = r_{l_i} < \ldots < r_{l_i+s_i} = t_{i+1}$.

Also,

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\Delta_i t \equiv t_{i+1} - t_i = r_{l_i+s_i} - r_{l_i} \equiv q_i \quad \Delta \equiv (n_{i+1} - n_i) \Delta.
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Therefore

$$
s_i \Delta \leq \sum_{l=l_i}^{l_i+s_i} (r_{l+1} - r_l) = q_i \Delta, \quad \text{i.e.} \ 1 \leq s_i \leq q_i \leq M_T.
$$
Moreover, given that times $t_i$ are chosen in a greedy fashion we also have for any $1 \leq R < s_i$

$$| \sum_{j=0}^{R-1} p_j | < Z, \quad 1 \leq |p_j| < 2Z, \quad 0 \leq j \leq s_i - 2, \quad |p_{s_i-1}| \leq (|m_i| + Z).$$
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Given that $m_i \delta_0 \equiv x(t_{i+1}) - x(t_i)$, hence $|m_i|\delta_0 \geq \delta$ is guaranteed for $i < N(x) - 1$ and so

$$|m_i| \geq Z, \ \text{for} \ 0 \leq i < N(x) - 1.$$
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$$|m_i| \geq Z, \ \text{for} \ 0 \leq i < N(x) - 1.$$

$$\Delta_i w \equiv w(x, t_{i+1}) - w(x, t_i) \equiv \sum_{j=0}^{s_i-1} |x(r_{i+j+1}) - x(r_{i+j})| \equiv$$

$$\delta_0 \sum_{j=0}^{s_i-1} |p_j| \equiv (j_{i+1} - j_i) \delta_0, \ \text{so} \ \ w(x, t_i) = \sum_{l=0}^{l_i-1} |x(r_{l+1}) - x(r_l)|.$$
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- $p_j$ the possible number of $\delta_0$ chart units in between two consecutive samples,
- $N(x)$ total number of portfolio rebalances in $[t_0, t_0 + T]$.

We will rely on $s_i \leq q_i$, that is we are taking an upper bound.
\( \delta_0 \)-Continuity, Jumps and In-Between Variation Changes

\[
S = s_0, \quad z = 4
\]
So we are observing upcrossings.
Definition of Trajectory Set

Actually we work with vector valued trajectories i.e. $S_i \rightarrow S_i \equiv (S_i, T_i, W_i)$, so we construct $S$ with elements $S \in S$, $S = \{S_i\}$. We just define a set of trajectories satisfying the previous constraints (we do introduce, later, external, empirical constraints on parameters). In other words: $(S_i, T_i, W_i)$ is defined by the obtained constraints obeyed by $(x(t_i), t_i, w(x, t_i))$. 
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Given a triple \((k_i, n_i, j_i)\) we allow for all possible \((k_{i+1}, n_{i+1}, j_{i+1})\) satisfying the previous constraints; for each such admissible triple we set:

\[
\Delta_i S \equiv (S_{i+1} - S_i) = (k_{i+1} - k_i)\delta_0 = m_i \delta_0,
\]

\[
\Delta_i W \equiv (W_{i+1} - W_i) = (j_{i+1} - j_i)\delta_0 = \sum_{j=0}^{s_i-1} |p_j|\delta_0,
\]

\[
\Delta_i T \equiv (T_{i+1} - T_i) = (n_{i+1} - n_i)\Delta = q_i \Delta, \text{ where } m_i, p_j \in \mathbb{Z}.
\]

\[1 \leq |p_j|, 1 \leq q_i \leq M_T, 1 \leq s_i \leq q_i, s_i, q_i \in \mathbb{N}\]
Moreover, define

\[ N(S) = i \text{ where } T_i = T, \]  

we remark that \( N(S) \) exists as \( \Delta_i T \geq \Delta \) for any \( i \geq 0 \).

For the case \( i < N(S) - 1 \) we also will require \( |m_i| \geq \frac{\delta}{\delta_0} = Z \). Also, for any \( 1 \leq R < s_i \),

\[
\sum_{j=0}^{s_i-1} p_j = m_i, \quad \sum_{j=0}^{R-1} p_j < Z, \quad |p_j| < 2Z, \quad 0 \leq j \leq s_i-2, \quad |p_{s_i-1}| \leq (|m_i| + Z).
\]
Empirical Restrictions Via Worst Case Estimated Parameters

- Let $N(x^*, [t, t + T])$ be the number of $\delta$-moves. Let $N_1$ be the minimum of $N(x^*, [t, t + T])$ over historical data $x^*(t)$. Let $N_2$ be the analogous maximum.
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- Let $\mathcal{N}_E \equiv \{(m, q) = \left(\frac{x^*(t_{i+1}) - x^*(t_i)}{\delta_0}, \frac{t_{i+1} - t_i}{\Delta}\right) : \text{collecting pairs over historical data } x^*(t)\}$
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- $N_1, N_2$ and $N_E$ are used in the construction of the trajectory set: we require $N_1 \leq N(S) \leq N_2$, $(\frac{\Delta_i S}{\delta_0}, \frac{\Delta_i T}{\Delta}) \in N_E$. 


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- $N_1, N_2$ and $\mathcal{N}_E$ are used in the construction of the trajectory set: we require $N_1 \leq N(S) \leq N_2, (\frac{\Delta_i S}{\delta_0}, \frac{\Delta_i T}{\Delta}) \in \mathcal{N}_E$.
- Similarly, for $0 \leq \rho \leq T$ define

$$V_\rho(x^*, [t, t + T]) \equiv \sum_{l=0}^{r_{l+1} \leq t + \rho} |x^*(r_{l+1}) - x^*(r_l)|,$$
Moreover, define

\[ V^*_\rho \equiv \bigcup_{t, t+T \subseteq T} \{ V_\rho(x^*, [t, t+T]) \}, \]

and require

\[ W_{T_i} \in V^*_{T_i}. \] (7)
Moreover, define

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and require

\[ W_{T_i} \in V^*_T. \]  \hspace{1cm} (7)

- A global constraint such as the above is what blocks a local extreme parameter (say \((m_{max}, q)\)) of being available at all nodes (if that is the case there will be a trajectory with too large a value of variation). Therefore, global constraints, in this case, tame the effect of worst case estimation on the price option bounds.
The two sources of data used for calibration and estimation are:

- A 6 month long stretch of hourly data ticks for Facebook Inc (FB). At the end of this period, call option ask prices were captured across a variety of strikes for comparison with an expiration of 9 days into the future.

- A 6 month long stretch of hourly ticks data for Biogen Inc (BIIB). At the end of this period, call option ask prices were captured across a variety of strikes for comparison with an expiration of 15 days into the future.
FB Price Bounds using observed conditional set
FB data (without using variation constraint).
Price Bounds using observed conditional set BIBB Data
BIIB observed conditional set without variation constraint
FB QV data
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One can trade risk and reward by removing some infrequent pairs \((m, q)\) or by making \(N_1\) larger or \(N_2\) smaller or by restricting the set \(V^*_\rho\).
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One can trade risk and reward by removing some infrequent pairs \((m, q)\) or by making \(N_1\) larger or \(N_2\) smaller or by restricting the set \(V_{\rho^*}\).

One can then bet on some rewards and on a risk that is worst case reliable.
Thank you!